

ON CONTROLLER-STOPPER PROBLEMS WITH JUMPS AND THEIR APPLICATIONS TO INDIFFERENCE PRICING OF AMERICAN OPTIONS

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ABSTRACT. We consider controller-stopper problems in which the controlled processes can have jumps. The global filtration is represented by the Brownian filtration, enlarged by the filtration generated by the jump process. We assume that the Brownian motion and jump process are independent, and there exists a probability density function for the jump times and marks. Under these assumptions, we decompose the global controller-stopper problem into controller-stopper problems with respect to the Brownian filtration, which are determined by a backward induction. We apply our decomposition method to indifference pricing of American options under multiple default risk. The backward induction leads to a system of reflected backward stochastic differential equations (RBSDEs). We show that there exists a solution to this RBSDE system and that the solution provides a characterization of the value function.

1. INTRODUCTION

The problem of pricing American options and the very closely related stochastic control problem of a controller and stopper either cooperating or playing a zero-sum game has been analyzed extensively for continuous processes. In particular, [10] considers the super-hedging problem; [11], [12], [13], and [2] consider the controller-stopper problems, and [16] resolves the indifference pricing problem using the results of [11]. We will consider the above problems in the presence of jumps in the state variables.

The stochastic control problems in the above setup can be solved by Hamilton-Jacobi-Bellman integro-differential equations in the Markovian setup, or by Reflected Backward Stochastic Differential Equations with jumps, generalizing the results of [8]. We prefer to use an alternative approach in which we convert the problem with jumps into a sequence of problems without jumps à la [1], which uses this result for linear pricing of American options, and [17] which uses this approach to solve indifference pricing problems for European-style optimal control problems with jumps under a conditional density hypothesis.

We assume that the underlying Brownian motion and jump process are independent. Following the set up in [9] and [17] we also assume that there are at most n jumps and that there exists a probability density for these jump times and the associated jump marks. In this jump-diffusion model, we give a decomposition of the controller-stopper problem into controller-stopper problems

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with respect to the Brownian filtration, which are determined by a backward induction. We apply this decomposition method to indifference pricing of American options under multiple jump risk, extending the results of [17]. The solution of this problem leads to a system of reflected backward stochastic differential equations (RBSDEs). We show that there exists a solution to this RBSDE system and the solution provides a characterization of the value function, which can be thought of as an extension of [7].

Our first result, see Theorem 2.1 and Proposition 2.3, is a decomposition result for stopping times of the global filtration (the filtration generated by the Brownian motion and jump times and marks). Next, in Section 3, we show that the expectation of an optional process with jumps can be computed by a backward induction, where each step is an expectation with respect to the Brownian filtration. In Section 4, we consider the controller-stopper problems with jumps and decompose the original problem into controller-stopper problems with respect to the Brownian filtration. Finally, we apply our decomposition result to obtain the indifference buying/selling price of American options with jump/default risk in Section 5 and characterize the optimal trading strategies and the optimal stopping times in Theorem 5.4 and Theorem 5.8, which resolves a saddle point problem.

In the rest of this section we will introduce the probabilistic setup and notation that we will use in the rest of the paper.

1.1. Probabilistic setup. Throughout this paper, we assume that the Brownian motion and jump process are independent. The “jump” here means the discontinuity of the underlying processes induced by sources other than the Brownian motion. The probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is formulated as follows:

Let $(\Omega_1, \mathbb{F}, \mathbb{P}_1)$ be the probability space corresponding to the Brownian motion, where Ω_1 is the canonical space, $\mathbb{F} = (\mathcal{F}_t)_{t=0}^\infty$ is the filtration generated by the Brownian motion, satisfying the usual conditions, and \mathbb{P}_1 is the Wiener measure. We assume that there exists at most n jumps. Define $\Delta_0 = \emptyset$ and

$$\Delta_k = \{(\theta_1, \dots, \theta_k) : 0 \leq \theta_1 \leq \dots \leq \theta_k\}, \quad k = 1, \dots, n,$$

which represents the space of first k jump times. For $k = 1, \dots, n$, let e_k be the k -th jump mark taking values in some Borel subset E of \mathbb{R}^d . Let \mathcal{D} be the filtration generated by the n jump times $(\zeta_1, \dots, \zeta_n)$ taking values in Δ_n , and their associated jump marks (ℓ_1, \dots, ℓ_n) taking values in E^n . I.e.,

$$\mathcal{D}_t = \vee_{i=1}^n (\sigma(1_{\{\zeta_i \leq s\}}, s \leq t) \vee \sigma(\ell_i 1_{\{\zeta_i \leq s\}}, s \leq t)).$$

We will also consider the filtration \mathcal{D}^k generated by the first k jump times and marks, i.e.,

$$\mathcal{D}_t^k = \vee_{i=1}^k (\sigma(1_{\{\zeta_i \leq s\}}, s \leq t) \vee \sigma(\ell_i 1_{\{\zeta_i \leq s\}}, s \leq t)), \quad k = 0, \dots, n.$$

The probability space for the first k jumps is denoted by $(\Omega_2, \mathcal{D}^k, \mathbb{P}_2)$, $k = 0, \dots, n$, where $\Omega_2 = \Delta_n \times E^n$ and \mathbb{P}_2 is some probability measure on Ω_2 . Let

$$\Omega := \Omega_1 \times \Omega_2, \quad \mathbb{P} := \mathbb{P}_1 \times \mathbb{P}_2, \quad \mathcal{G}_t^k = \mathcal{F}_t \otimes \mathcal{D}_t^k, \quad k = 0, \dots, n.$$

Denote by $\mathbb{G}^k = (\mathcal{G}_t^k)_{t=0}^\infty$ for $k = 0, \dots, n$, and $\mathbb{G} = \mathbb{G}^n$. Then $(\Omega, \mathbb{G}^k, \mathbb{P})$ is the probability space including at most the first k jumps, $k = 0, \dots, n$. Let $(\Omega, \mathbb{G}, \mathbb{P}) = (\Omega, \mathbb{G}^n, \mathbb{P})$ which we refer to as the global probability space. Note that for $k = 0, \dots, n$, we may characterize each element in Ω as $(\omega_1, \theta_1, \dots, \theta_k, e_1, \dots, e_k)$, when the random variable we consider is \mathcal{G}_∞^k -measurable, where $\mathcal{G}_\infty^k = \cup_{t=0}^\infty \mathcal{G}_t^k$, see page 76 in [4].

Next we will introduce some notation that will be used in the rest of the paper.

1.2. Notation.

- For any $(\theta_1, \dots, \theta_k) \in \Delta_k$, $(\ell_1, \dots, \ell_k) \in E^k$, we denote by

$$\boldsymbol{\theta}_k = (\theta_1, \dots, \theta_k), \quad \boldsymbol{\ell}_k = (\ell_1, \dots, \ell_k), \quad k = 1, \dots, n.$$

We also denote by $\boldsymbol{\zeta}_k = (\zeta_1, \dots, \zeta_k)$, and $\boldsymbol{\ell}_k = (\ell_1, \dots, \ell_k)$. From now on, for $k = 1, \dots, n$, we use $\theta_k, \boldsymbol{\theta}_k, e_k, \mathbf{e}_k$ to represent given fixed numbers or vectors, and $\zeta_k, \boldsymbol{\zeta}_k, \ell_k, \boldsymbol{\ell}_k$ to represent random jump times or marks.

- $\mathcal{P}_{\mathbb{F}}$ is the σ -algebra of \mathbb{F} -predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e., the σ -algebra generated by the left-continuous \mathbb{F} -adapted processes.
- $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is the set of indexed \mathbb{F} -predictable processes $Z^k(\cdot)$, i.e., the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \rightarrow Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable, for $k = 1, \dots, n$. We also denote $\mathcal{P}_{\mathbb{F}}$ as $\mathcal{P}_{\mathbb{F}}(\Delta_0, E^0)$.
- $\mathcal{O}_{\mathbb{F}}$ is the σ -algebra of \mathbb{F} -optional measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e., the σ -algebra generated by the left-continuous \mathbb{F} -adapted processes.
- $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ is the set of indexed \mathbb{F} -adapted processes $Z^k(\cdot)$, i.e., the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \rightarrow Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable, for $k = 1, \dots, n$. We also denote $\mathcal{O}_{\mathbb{F}}$ as $\mathcal{O}_{\mathbb{F}}(\Delta_0, E^0)$.
- For any \mathcal{G}_∞^k -measurable random variable X , we sometimes denote it as $X = X(\omega_1, \omega_2) = X(\omega_1, \boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) = X(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k)$. Given $\boldsymbol{\zeta}_k = \boldsymbol{\theta}_k$, $\boldsymbol{\ell}_k = \mathbf{e}_k$, we denote X as $X = X(\omega_1, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = X(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. Similar notations apply for any \mathbb{G}^k -adapted process $(Z_t)_{t \geq 0}$ and its stopped version Z_τ , where τ is a \mathbb{G}^k -stopping time.
- For $T \in [0, \infty]$, $\Delta_k(T) := \Delta_k \cap [0, T]^k$.
- $\mathcal{S}_c^\infty[t, T] := \left\{ Y : \mathbb{F}\text{-adapted continuous, } \|Y\|_{\mathcal{S}_c^\infty[t, T]} := \operatorname{ess\,sup}_{(s, \omega) \in [t, T] \times \Omega} |Y_s(\omega)| < \infty \right\}$.
- $\mathcal{S}_c^\infty(\Delta_k(T), E^k) := \left\{ Y^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k) : Y^k \text{ is continuous, and } \|Y^k\|_{\mathcal{S}_c^\infty(\Delta_k(T), E^k)} := \sup_{(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k} \|Y^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\|_{\mathcal{S}_c^\infty[\boldsymbol{\theta}_k, T]} < \infty \right\}, \quad k = 0, \dots, n.$
- $\mathbf{L}_W^2[t, T] := \left\{ Z : \mathbb{F}\text{-predictable, } \mathbb{E} \left[\int_t^T |Z_s|^2 ds \right] < \infty \right\}$.
- $\mathbf{L}_W^2(\Delta_k(T), E^k) := \left\{ Z^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k) : \mathbb{E} \left[\int_{\boldsymbol{\theta}_k}^T |Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 dt \right] < \infty, \quad \forall (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k \right\}, \quad k = 0, \dots, n.$
- $\mathbf{A}[t, T] := \left\{ K : \mathbb{F}\text{-adapted continuous increasing, } K_t = 0, \mathbb{E} K_T^2 < \infty \right\}, \quad k = 0, \dots, n.$
- $\mathbf{A}(\Delta_k(T), E^k) := \left\{ K^k : \forall (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k, \quad K^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \mathbf{A}[\boldsymbol{\theta}_k, T] \right\}, \quad k = 0, \dots, n.$

- We use $\mathbf{eq}(H, f)_{s \leq t \leq T}$ to represent the RBSDE

$$\begin{cases} Y_t = H_T - \int_t^T f(r, Y_r, Z_r) dr + \int_t^T Z_r dW_r + (K_T - K_t), & s \leq t \leq T, \\ Y_t \geq H_t, & s \leq t \leq T, \\ \int_t^T (Y_t - H_t) dK_t = 0, & s \leq t \leq T, \end{cases}$$

and $\mathbf{EQ}(\mathcal{H}, \mathfrak{f})_{s \leq t \leq T}$ to represent the RBSDE

$$\begin{cases} \mathcal{Y}_t = \mathcal{H}_T + \int_t^T \mathfrak{f}(r, \mathcal{Y}_r, \mathcal{Z}_r) dr - \int_t^T \mathcal{Z}_r dW_r + (\mathcal{K}_T - \mathcal{K}_t), & s \leq t \leq T, \\ \mathcal{Y}_t \geq \mathcal{H}_t, & s \leq t \leq T, \\ \int_t^T (\mathcal{Y}_t - \mathcal{H}_t) d\mathcal{K}_t = 0, & s \leq t \leq T. \end{cases}$$

2. DECOMPOSITION OF \mathbb{G} -STOPPING TIMES

Theorem 2.1 and Proposition 2.3 are the main results of this section.

Theorem 2.1. τ is a \mathbb{G} -stopping time if and only if it has the decomposition:

(2.1)

$$\tau = \tau^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k(\zeta_k, \ell_k) 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tau^n(\zeta_n, \ell_n) 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}},$$

for some (τ^0, \dots, τ^n) , where τ^0 is an \mathbb{F} -stopping time, and $\tau^k(\zeta_k, \ell_k)$ is a \mathbb{G}^k -stopping time satisfying

(2.2)

$$\tau^k(\zeta_k, \ell_k) \geq \zeta_k, \quad k = 1, \dots, n.$$

Proof. If τ has the decomposition (2.1), then

$$\begin{aligned} \{\tau \leq t\} &= \left(\{\tau^0 < \zeta_1\} \cap \{\tau^0 \leq t\} \right) \bigcup_{k=1}^{n-1} \left(\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \cap \{\tau^k \leq t\} \right) \\ &\quad \cup \left(\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\} \cap \{\tau^n \leq t\} \right). \end{aligned}$$

For $k = 1, \dots, n$, since $\{\tau^k < \zeta_{k+1}\} \in \mathcal{G}_{\tau^k}$, and

$$\{\tau^{i-1} \geq \zeta_i\} \in \mathcal{G}_{\zeta_i} \subset \mathcal{G}_{\zeta_k} \subset \mathcal{G}_{\tau^k}, \quad i = 1, \dots, k,$$

we have

$$\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \cap \{\tau^k \leq t\} \in \mathcal{G}_t.$$

Similarly we can show $\{\tau^0 < \zeta_1\} \cap \{\tau^0 \leq t\} \in \mathcal{G}_t$ and

$$\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\} \cap \{\tau^n \leq t\} \in \mathcal{G}_t.$$

If τ is a \mathbb{G} -stopping time, we will proceed in 3 steps to show that it has the decomposition (2.1).

Step 1: We will show that for any discretely valued \mathbb{G} -stopping time

$$\tau = \sum_{1 \leq i \leq \infty} a_i 1_{A_i},$$

where $0 \leq a_1 < a_2 < \dots < a_\infty = \infty$ and $(A_i \in \mathcal{G}_{a_i})_{1 \leq i \leq \infty}$ is a partition of Ω , there exists a \mathbb{G}^k -stopping time $\tau^k = \tau^k(\zeta_k, \ell_k)$, such that

$$(2.3) \quad \tau 1_{\{\tau < \zeta_{k+1}\}} = \tau^k 1_{\{\tau < \zeta_{k+1}\}} \quad \text{and} \quad \{\tau < \zeta_{k+1}\} = \{\tau^k < \zeta_{k+1}\},$$

for $k = 0, \dots, n-1$. First, we have

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1 \leq i \leq \infty} (\{\tau < \zeta_{k+1}\} \cap \{A_i\}) = \bigcup_{1 \leq i \leq \infty} (\{a_i < \zeta_{k+1}\} \cap \{A_i\}).$$

To complete Step 1, we need the following lemma:

Lemma 2.2. *For $i = 1, \dots, \infty$, and $A_i \in \mathcal{G}_{a_i}$, there exists $\tilde{A}_i \in \mathcal{G}_{a_i}^k$, such that*

$$(2.4) \quad \{a_i < \zeta_{k+1}\} \cap \tilde{A}_i = \{a_i < \zeta_{k+1}\} \cap A_i.$$

Moreover, $(\tilde{A}_i)_{1 \leq i \leq \infty}$ can be chosen to be mutually disjoint.

Proof of Lemma 2.2. Since for $j \geq k+1$,

$$\begin{aligned} & (\sigma(1_{\{\zeta_j \leq s\}}, s \leq a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \leq s\}}, s \leq a_i)) \cap \{a_i < \zeta_{k+1}\} \\ &= \sigma\left(\{\zeta_j \leq s\}, (\{\ell \in C\} \cap \{\zeta_j \leq t\}) \cup \{\zeta_j > t\}, s, t \leq a_i, C \in \mathcal{B}(E)\right) \cap \{a_i < \zeta_{k+1}\} \\ &= \sigma\left(\{\zeta_j \leq s\} \cap \{a_i < \zeta_{k+1}\}, ((\{\ell \in C\} \cap \{\zeta_j \leq t\}) \cup \{\zeta_j > t\}) \cap \{a_i < \zeta_{k+1}\}, \right. \\ & \quad \left. s, t \leq a_i, C \in \mathcal{B}(E)\right) \\ &= \{\emptyset, \{a_i < \zeta_{k+1}\}\}, \end{aligned}$$

we have

$$\begin{aligned} & \mathcal{G}_{a_i} \cap \{a_i < \zeta_{k+1}\} \\ &= \left(\mathcal{F}_{a_i} \otimes \left(\bigvee_{j=1}^n (\sigma(1_{\{\zeta_j \leq s\}}, s \leq a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \leq s\}}, s \leq a_i)) \right) \right) \cap \{a_i < \zeta_{k+1}\} \\ &= \left(\mathcal{F}_{a_i} \otimes \left(\bigvee_{j=1}^n (\sigma(1_{\{\zeta_j \leq s\}}, s \leq a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \leq s\}}, s \leq a_i)) \cap \{a_i < \zeta_{k+1}\} \right) \right) \\ &= \left(\mathcal{F}_{a_i} \otimes \left(\bigvee_{j=1}^k (\sigma(1_{\{\zeta_j \leq s\}}, s \leq a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \leq s\}}, s \leq a_i)) \cap \{a_i < \zeta_{k+1}\} \right) \right) \\ &= \left(\mathcal{F}_{a_i} \otimes \left(\bigvee_{j=1}^k (\sigma(1_{\{\zeta_j \leq s\}}, s \leq a_i) \vee \sigma(\ell_j 1_{\{\zeta_j \leq s\}}, s \leq a_i)) \right) \right) \cap \{a_i < \zeta_{k+1}\} \\ &= \mathcal{G}_{a_i}^k \cap \{a_i < \zeta_{k+1}\}, \end{aligned}$$

which proves the existence result in Lemma 2.2. Now suppose $(\bar{A}_i \in \mathcal{G}_{a_i}^k)_{1 \leq i \leq \infty}$ are the sets such that (2.4) holds. Define $\tilde{A}_1 = \bar{A}_1$, $\tilde{A}_\infty = \emptyset$, and

$$\tilde{A}_{m+1} = \bar{A}_{m+1} \setminus \bigcup_{j=1}^m \bar{A}_j, \quad m = 1, 2, \dots$$

Since for $i \neq j$, $(\bar{A}_i \cap \{a_i < \zeta_{k+1}\}) \cap (\bar{A}_j \cap \{a_j < \zeta_{k+1}\}) = \emptyset$, we have for $m = 1, 2, \dots$,

$$\begin{aligned}
\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} &\supset \tilde{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} \\
&= (\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}) \setminus \bigcup_{j=1}^m (\bar{A}_j \cap \{a_{m+1} < \zeta_{k+1}\}) \\
&\supset (\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}) \setminus \bigcup_{j=1}^m (\bar{A}_j \cap \{a_j < \zeta_{k+1}\}) \\
&= (\bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}).
\end{aligned}$$

Therefore, $\tilde{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\} = \bar{A}_{m+1} \cap \{a_{m+1} < \zeta_{k+1}\}$, and thus $(\tilde{A}_i \in \mathcal{G}_{a_i}^k)_{1 \leq i \leq \infty}$ are the disjoint sets such that (2.4) holds. This completes the proof of Lemma 2.2. \square

Now let us continue with the proof of Theorem 2.1. From Lemma 2.2, we have

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1 \leq i \leq \infty} (\{a_i < \zeta_{k+1}\} \cap \tilde{A}_i),$$

where $(\tilde{A}_i \in \mathcal{G}_{a_i}^k)_{1 \leq i \leq \infty}$ are disjoint sets such that (2.4) holds. Define \mathbb{G}^k -stopping time

$$\tau^k = \sum_{1 \leq i \leq \infty} a_i 1_{\tilde{A}_i}.$$

Since

$$\tilde{A}_i \cap \{\tau < \zeta_{k+1}\} = \tilde{A}_i \cap \bigcup_{1 \leq j \leq \infty} (\{a_j < \zeta_{k+1}\} \cap \tilde{A}_j) = \{a_i < \zeta_{k+1}\} \cap \tilde{A}_i = \{\tau < \zeta_{k+1}\} \cap A_i,$$

we have

$$\tau^k 1_{\{\tau < \zeta_{k+1}\}} = \sum_{1 \leq i \leq \infty} a_i 1_{\tilde{A}_i \cap \{\tau < \zeta_{k+1}\}} = \sum_{1 \leq i \leq \infty} a_i 1_{A_i \cap \{\tau < \zeta_{k+1}\}} = \tau 1_{\{\tau < \zeta_{k+1}\}}.$$

Also,

$$\{\tau < \zeta_{k+1}\} = \bigcup_{1 \leq i \leq \infty} (\{a_i < \zeta_{k+1}\} \cap A_i) = \bigcup_{1 \leq i \leq \infty} (\{a_i < \zeta_{k+1}\} \cap \tilde{A}_i) = \{\tau^k < \zeta_{k+1}\}.$$

Step 2: We will show that for any \mathbb{G} -stopping time τ , there exists a \mathbb{G}^k -stopping time τ^k , such that (2.3) holds. Define the \mathbb{G} -stopping times

$$\tau_m := \sum_{j=0}^{\infty} \frac{j+1}{2^m} \cdot 1_{\{\frac{j}{2^m} \leq \tau < \frac{j+1}{2^m}\}} + \infty \cdot 1_{\{\tau = \infty\}}, \quad m = 1, 2, \dots$$

By Step 1, there exists a \mathbb{G}^k -stopping time τ_m^k , such that

$$(2.5) \quad \tau_m^k 1_{\{\tau_m < \zeta_{k+1}\}} = \tau_m 1_{\{\tau_m < \zeta_{k+1}\}} \quad \text{and} \quad \{\tau_m < \zeta_{k+1}\} = \{\tau_m^k < \zeta_{k+1}\}.$$

Define $\tau^k := \limsup_{m \rightarrow \infty} \tau_m^k$. Since $\tau_m \searrow \tau$, by taking “lim sup” on both side of (2.5), we have (2.3).

Step 3: From Step 2, we know that for any \mathbb{G} -stopping time τ , there exists $\tau^0, \tau^1, \dots, \tau^{n-1}$ being $\mathbb{F}, \mathbb{G}^1, \dots, \mathbb{G}^{n-1}$ -stopping times respectively, such that (2.3) holds, for $k = 0, \dots, n-1$. Let $\tau^n := \tau$, then we have

$$\begin{aligned}
\tau &= \tau 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tau 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \tau 1_{\{\zeta_n \leq \tau\}} \\
&= \tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \tau^n 1_{\{\zeta_n \leq \tau\}} \\
&= \tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau \geq \zeta_1\} \cap \dots \cap \{\tau \geq \zeta_k\} \cap \{\tau < \zeta_{k+1}\}} + \tau^n 1_{\{\tau \geq \zeta_1\} \cap \dots \cap \{\tau \geq \zeta_n\}} \\
&= \tau^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tau^n 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}}.
\end{aligned}$$

We will modify the decomposition so that it satisfies (2.2). For $k = 1, \dots, n$, define \mathbb{G}^k -stopping time

$$\tilde{\tau}^k = \begin{cases} \tau^k, & \tau^k \geq \zeta_k, \\ \zeta_k, & \tau^k < \zeta_k. \end{cases}$$

and let $\tilde{\tau}^0 := \tau^0$. Then for $k = 1, \dots, n$, $\tilde{\tau}^k \geq \zeta_k$, and

$$\{\tilde{\tau}^k < \zeta_{k+1}\} = \{\tau^k < \zeta_{k+1}\} = \{\tau < \zeta_{k+1}\}, \quad k = 0, \dots, n-1.$$

For $k = 1, \dots, n-1$, since $\{\zeta_k \leq \tau < \zeta_{k+1}\} \subset \{\tau = \tau^k\}$, we have

$$\begin{aligned}
&\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \\
&= \{\zeta_k \leq \tau < \zeta_{k+1}\} = \{\zeta_k \leq \tau < \zeta_{k+1}\} \cap \{\tau = \tau^k\} \subset \{\tau^k \geq \zeta_k\}.
\end{aligned}$$

Also $\{\tau \geq \zeta_n\} \subset \{\tau = \tau^n\}$ implies

$$\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\} = \{\tau \geq \zeta_n\} = \{\tau \geq \zeta_n\} \cap \{\tau = \tau^n\} \subset \{\tau^n \geq \zeta_n\}.$$

Therefore, we have

$$\begin{aligned}
\tau &= \tau^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tau^n 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}} \\
&= \tilde{\tau}^0 1_{\{\tau^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}} \\
&= \tilde{\tau}^0 1_{\{\tilde{\tau}^0 < \zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\tilde{\tau}^0 \geq \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{k-1} \geq \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\tilde{\tau}^0 \geq \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{n-1} \geq \zeta_n\}}.
\end{aligned}$$

This ends the proof of Theorem 2.1. \square

In the rest of the paper, we will use the notation $\tau \sim (\tau^0, \dots, \tau^n)$ for the \mathbb{G} -stopping time τ if it has the decomposition from (2.1). The next result shows that the decomposition of τ in (2.1) is unique, in the sense that the terms in the sum of τ 's representation are the same even for different

(τ^0, \dots, τ^n) 's in the representation. (Note that one can modify the stopping times τ^i after the jump times ζ_{i+1} .)

Proposition 2.3. *Let $\tau \sim (\tau^0, \dots, \tau^n)$ be a \mathbb{G} -stopping time. Then $\{\tau^0 < \zeta_1\} = \{\tau < \zeta_1\}$, $\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} = \{\zeta_k \leq \tau < \zeta_{k+1}\}$ for $k = 1, \dots, n-1$, and $\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\} = \{\zeta_n \leq \tau\}$. Therefore,*

$$\tau = \tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tau^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \tau^n 1_{\{\zeta_n \leq \tau\}}.$$

Proof. Let $A_0 := \{\tau^0 < \zeta_1\}$, $A_n := \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}$, and

$$A_k := \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}, \quad k = 1, \dots, n-1.$$

Let $B_0 := \{\tau < \zeta_1\}$, $B_n := \{\zeta_n \leq \tau\}$, and $B_k := \{\zeta_k \leq \tau < \zeta_{k+1}\}$, $k = 1, \dots, n-1$. In the set A_i , we have $\tau = \tau^i$, which implies $\zeta_i \leq \tau < \zeta_{i+1}$, and thus $A_i \subset B_i$, for $i = 1, \dots, n-1$. Similarly, $A_0 \subset B_0$ and $A_n \subset B_n$. Since $(A_i)_{i=0}^n$ and $(B_i)_{i=0}^n$ are mutually disjoint respectively, and $\Omega = \bigcup_{i=0}^n A_i = \bigcup_{i=0}^n B_i$, we have $A_i = B_i$, $i = 0, \dots, n$. \square

The last proposition generalizes the decomposition result given in Theorem (A2.3) of [5] on page 261 (also see Theorem T33 of [3] on page 308) from the stopping times of piecewise deterministic Markov processes to the stopping times of jump diffusions.

Proposition 2.4. *Let $T > 0$ be a constant. τ is an \mathbb{G} -stopping time satisfying $\tau \leq T$ if and only if τ has the decomposition (2.1), with $\tau^0 \leq T$ and $\{\zeta_k \leq T\} = \{\tau^k \leq T\}$, $k = 1, \dots, n$.*

Proof. If τ has the decomposition, then on the set $\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\}$, we have

$$T \geq \tau^0 \geq \zeta_1 \Rightarrow T \geq \tau^1 \Rightarrow T \geq \zeta_2 \Rightarrow \dots \Rightarrow T \geq \tau^{k-1} \Rightarrow T \geq \zeta_k \Rightarrow T \geq \tau^k,$$

For $k = 1, \dots, n$. Thus $\tau \leq T$.

Conversely, let $\tau \sim (\tau^0, \dots, \tau^n)$ be a \mathbb{G} -stopping time satisfying $\tau \leq T$. Let $\tilde{\tau}^0 := \tau^0$, and

$$\tilde{\tau}^k := \begin{cases} \tau^k \wedge T, & \zeta_k \leq T, \\ \tau^k, & \zeta_k > T. \end{cases}$$

for $k = 0, \dots, n$. It can be shown that $\tilde{\tau}^k$ is a \mathbb{G}^k -stopping time. Then for $k = 1, \dots, n-1$,

$$\zeta_k \leq \tau < \zeta_{k+1} \Rightarrow \tau^k = \tau \leq T \Rightarrow \tilde{\tau}^k = \tau^k.$$

Similarly, $\zeta_n \leq \tau \Rightarrow \tilde{\tau}^n = \tau^n$. Therefore,

$$\tau = \tilde{\tau}^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \tilde{\tau}^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \tilde{\tau}^n 1_{\{\zeta_n \leq \tau\}}.$$

Easy to see $\tilde{\tau}^k \geq \zeta_k$ and $\{\zeta_k \leq T\} = \{\tilde{\tau}^k \leq T\}$, $k = 1, \dots, n$. It remains to show $A_i = B_i$, $i = 0, \dots, n$, where $A_0 := \{\tau^0 < \zeta_1\}$, $A_n := \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{n-1} \geq \zeta_n\}$,

$$A_k := \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}, \quad k = 0, \dots, n-1,$$

and $B_0 := \{\tilde{\tau}^0 < \zeta_1\}$, $B_n := \{\tilde{\tau}^0 \geq \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{n-1} \geq \zeta_n\}$,

$$B_k := \{\tilde{\tau}^0 \geq \zeta_1\} \cap \dots \cap \{\tilde{\tau}^{k-1} \geq \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\}, \quad k = 0, \dots, n-1.$$

Easy to see $A_0 = B_0$ and $A_n \supset B_n$. Now for $k = 1, \dots, n-1$,

$$\begin{aligned} & \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} \\ & \subset \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \left(\{\tau^k < \zeta_{k+1}\} \cup \{T < \zeta_{k+1}\} \right). \end{aligned}$$

Since

$$\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{T < \zeta_{k+1}\} \cap \{\tau^k \geq \zeta_{k+1}\} = \emptyset,$$

we have

$$\{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{T < \zeta_{k+1}\} \subset \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\}.$$

Hence, for $k = 1, \dots, n-1$,

$$\begin{aligned} B_k & \subset \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tilde{\tau}^k < \zeta_{k+1}\} \\ & = \{\tau^0 \geq \zeta_1\} \cap \dots \cap \{\tau^{k-1} \geq \zeta_k\} \cap \{\tau^k < \zeta_{k+1}\} = A_k \end{aligned}$$

Since $\bigcup_{k=0}^n A_k = \bigcup_{k=0}^n B_k = \Omega$, and $(A_k)_{k=0}^n$ and $(B_k)_{k=0}^n$ are mutually disjoint respectively, we have $A_k = B_k$, $k = 0, \dots, n$. \square

3. DECOMPOSITION OF EXPECTATIONS OF \mathbb{G} -OPTIONAL PROCESSES

The main result in this section is Theorem 3.3, which shows that the expectation of a stopped \mathbb{G} -optional process can be calculated using a backward induction, where each step is an expectation with respect to the Brownian filtration.

Standing Assumption: For the rest of the paper, we assume there exists a probability density function α , such that

$$\begin{aligned} (3.1) \quad \mathbb{P}_2[(\zeta_1, \dots, \zeta_n, \ell_1, \dots, \ell_n) \in d\theta_1 \dots d\theta_n de_1 \dots de_n] \\ = \alpha(\theta_1, \dots, \theta_n, e_1, \dots, e_n) d\theta_1 \dots d\theta_n \eta(de_1) \dots \eta(de_n), \end{aligned}$$

where $d\theta_k$ is the Lebesgue measure, and $\eta(de_k)$ is some probability measure which may depend on $(\theta_{k-1}, \mathbf{e}_{k-1})$ (e.g., transition kernel), for $k = 1, \dots, n$.

Following [17], let us set $\alpha_t^n(\theta_n, \mathbf{e}_n) = \alpha(\theta_n, \mathbf{e}_n)$, and

$$(3.2) \quad \alpha_t^k(\theta_k, \mathbf{e}_k) = \int_E \int_t^\infty \alpha_t^{k+1}(\theta_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}) d\theta_{k+1} \eta(de_{k+1}), \quad k = 0, \dots, n-1.$$

Note that $\alpha = 0$ when $\theta_1, \dots, \theta_n$ are not in an ascending order. As a result, for $k = 0, \dots, n-1$,

$$\alpha_t^k(\theta_k, \mathbf{e}_k) = \int_{E^k} \int_t^\infty \int_{\theta_{k+1}}^\infty \dots \int_{\theta_{n-1}}^\infty \alpha(\theta_n, \mathbf{e}_n) d\theta_n \dots d\theta_{k+1} \eta(de_n) \dots \eta(de_{k+1}).$$

Hence $\mathbb{P}[\zeta_1 > t] = \alpha_t^0$, and for $k = 1, \dots, n-1$,

$$\mathbb{P}[\zeta_{k+1} > t] = \int_{E^k} \int_{\Delta_k} \alpha_t^k(\theta_1, \dots, \theta_k, e_1, \dots, e_k) d\theta_1 \dots d\theta_k \eta(de_1) \dots \eta(de_k).$$

Therefore, α^k can be interpreted as the survival density of ζ_{k+1} .

Let us recall the following lemma from [17].

Lemma 3.1. *Any process $Z = (Z_t)_{t \geq 0}$ is \mathbb{G} -optional if and only if it has the decomposition:*

$$(3.3) \quad Z_t = Z_t^0 1_{\{t < \zeta_1\}} + \sum_{k=1}^{n-1} Z_t^k(\zeta_k, \ell_k) 1_{\{\zeta_k \leq t < \zeta_{k+1}\}} + Z_t^n(\zeta_n, \ell_n) 1_{\{\zeta_n \leq t\}},$$

for some $Z^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, for $k = 0, \dots, n$. A similar decomposition result holds for any \mathbb{G} -predictable process.

We will use the notation $Z \sim (Z^0, \dots, Z^n)$ for the \mathbb{G} -optional (resp. predictable) process Z from the decomposition (3.3). Let $Z \sim (Z^0, \dots, Z^n)$ be a \mathbb{G} -optional process, and $\tau \sim (\tau^0, \dots, \tau^n)$ be a \mathbb{G} -stopping time. Then from Lemma 3.1 and Proposition 2.3, Z_τ has the decomposition:

$$(3.4) \quad Z_\tau = Z_\tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} Z_\tau^k 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + Z_\tau^n 1_{\{\zeta_n \leq \tau\}}.$$

The following lemma will be used for the rest of the paper:

Lemma 3.2. $\tau^k(\zeta_k, \ell_k)$ is a \mathbb{G}^k -stopping time satisfying $\tau^k \geq \zeta_k$ if and only if for any fixed $(\theta_k, \mathbf{e}_k) \in \Delta_k \times E^k$, $\tau^k(\theta_k, \mathbf{e}_k)$ is an \mathbb{F} -stopping time satisfying $\tau^k(\theta_k, \mathbf{e}_k) \geq \theta_k$ and $\tau^k(\theta_k, \mathbf{e}_k)$ is measurable with respect to (θ_k, \mathbf{e}_k) .

Proof. If $\tau^k(\zeta_k, \ell_k)$ is a \mathbb{G}^k -stopping time, then for any $t \geq 0$, $\{\tau^k(\zeta_k, \ell_k) \leq t\} \in \mathcal{F}_t \otimes \mathcal{D}_t^k$. Then for its projection onto $(\zeta_k, \ell_k) = (\theta_k, \mathbf{e}_k)$, we have $\{\omega_1 : \tau^k(\omega_1, \theta_k, \mathbf{e}_k) \leq t\} \in \mathcal{F}_t$. Conversely, if $\tau^k(\theta_k, \mathbf{e}_k)$ is an \mathbb{F} -stopping time satisfying $\tau^k(\theta_k, \mathbf{e}_k) \geq \theta_k$ and is measurable with respect to (θ_k, \mathbf{e}_k) , then $1_{\{\tau^k(\theta_k, \mathbf{e}_k) \leq t\}} \cdot 1_{\{\theta_k \leq t\}} \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$. By Lemma 3.1 (here $n = k$), $1_{\{\tau^k(\zeta_k, \ell_k) \leq t\}} = 1_{\{\tau^k(\zeta_k, \ell_k) \leq t\}} \cdot 1_{\{\zeta_k \leq t\}}$ is a \mathbb{G}^k -optional process. Then $\{\tau^k(\zeta_k, \ell_k) \leq t\} = \{1_{\{\tau^k(\zeta_k, \ell_k) \leq t\}} = 1\} \in \mathcal{G}_t^k$. Hence, $\tau^k(\zeta_k, \ell_k)$ is a \mathbb{G}^k -stopping time. \square

The following theorem is the main result of this section.

Theorem 3.3. *Let $Z \sim (Z^0, \dots, Z^n)$ be a nonnegative (or bounded) \mathbb{G} -optional process, and $\tau \sim (\tau^0, \dots, \tau^n)$ be a finite \mathbb{G} -stopping time satisfying $\tau \leq T$, where $T \in [0, \infty]$ is a constant. The expectation $\mathbb{E}[Z_\tau]$ can be computed by a backward induction as*

$$\mathbb{E}[Z_\tau] = J_0,$$

where J_0, \dots, J_n are given by

$$(3.5) \quad J_n(\theta_n, \mathbf{e}_n) = \mathbb{E}\left[Z_{\tau^n}^n \alpha(\theta_n, \mathbf{e}_n) \middle| \mathcal{F}_{\theta_n}\right], \quad (\theta_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n,$$

$$(3.6) \quad J_k(\theta_k, \mathbf{e}_k) = \mathbb{E}\left[Z_{\tau^k}^k \alpha_{\tau^k}^k(\theta_k, \mathbf{e}_k) + \int_{\theta_k}^{\tau^k(\theta_k, \mathbf{e}_k) \wedge T} \int_E J_{k+1}(\theta_k, \theta_{k+1}, \mathbf{e}_k, \mathbf{e}_{k+1}) \eta(d\mathbf{e}_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k}\right],$$

$(\theta_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$.

Proof. For the sake of simplicity, let us assume $n = 2$. Using (3.5) and (3.6), plugging J_2 into J_1 , and then J_1 into J_0 , we obtain

$$\begin{aligned} J_0 &= \mathbb{E}\left[Z_{\tau^0}^0 \alpha_{\tau^0}^0\right] + \mathbb{E}\left[\int_0^{\tau^0 \wedge T} \int_E \mathbb{E}\left[Z_{\tau^1(\theta_1, e_1)}^1 \cdot \alpha_{\tau^1(\theta_1, e_1)}^1 \middle| \mathcal{F}_{\theta_1}\right] \eta(de_1) d\theta_1\right] \\ &+ \mathbb{E}\left[\int_0^{\tau^0 \wedge T} \int_E \mathbb{E}\left[\int_0^{\tau^1(\theta_1, e_1) \wedge T} \int_E \mathbb{E}\left[Z_{\tau^2(\theta_1, \theta_2, e_1, e_2)}^2 \cdot \alpha \middle| \mathcal{F}_{\theta_2}\right] \eta(de_2) d\theta_2 \middle| \mathcal{F}_{\theta_1}\right] \eta(de_1) d\theta_1\right]. \end{aligned}$$

On the right side of the equation above, let us denote the first term by I, the second term by II, and the third term by III. We can show that

$$\begin{aligned} \text{I} &= \mathbb{E}\left[\int_{E^2} \int_{\Delta_2} Z_{\tau^0}^0 \cdot 1_{\{\theta_1 > \tau^0\}} \cdot \alpha(\theta_1, \theta_2, e_1, e_2) d\theta_1 d\theta_2 \eta(de_1) \eta(de_2)\right], \\ \text{II} &= \mathbb{E}\left[\int_{E^2} \int_{\Delta_2} Z_{\tau^1(\theta_1, e_1)}^1 \cdot 1_{\{\theta_1 \leq T\}} \cdot 1_{\{\tau^0 \geq \theta_1\} \cap \{\tau^1(\theta_1, e_1) < \theta_2\}} \cdot \alpha d\theta_1 d\theta_2 \eta(de_1) \eta(de_2)\right], \\ \text{III} &= \mathbb{E}\left[\int_{E^2} \int_{\Delta_2} Z_{\tau^2(\theta_1, \theta_2, e_1, e_2)}^2 \cdot 1_{\{\theta_1, \theta_2 \leq T\}} \cdot 1_{\{\tau^0 \geq \theta_1\} \cap \{\tau^1(\theta_1, e_1) \geq \theta_2\}} \cdot \alpha d\theta_1 d\theta_2 \eta(de_1) \eta(de_2)\right]. \end{aligned}$$

For fixed $(\theta_1, \theta_2, e_1, e_2) \in \Delta_2 \times E^2$, from Proposition 2.3, we have $\{\tau^0 \geq \theta_1\} \cap \{\tau^1 < \theta_2\} = \{\theta_1 \leq \tau < \theta_2\} \subset \{\theta_1 \leq T\}$, and $\{\tau^0 \geq \theta_1\} \cap \{\tau^1 \geq \theta_2\} = \{\theta_2 \leq \tau\} \subset \{\theta_1, \theta_2 \leq T\}$. Hence,

$$\begin{aligned} Z_\tau(\theta_1, \theta_2, e_1, e_2) &= Z_{\tau^0}^0 \cdot 1_{\{\tau^0 < \theta_1\}} + Z_{\tau^1}^1 \cdot 1_{\{\tau^0 \geq \theta_1\}} \cdot 1_{\{\tau^1 < \theta_2\}} + Z_{\tau^2}^2 \cdot 1_{\{\tau^0 \geq \theta_1\}} \cdot 1_{\{\tau^1 \geq \theta_2\}} \\ &= Z_{\tau^0}^0 \cdot 1_{\{\tau^0 < \theta_1\}} + Z_{\tau^1}^1 \cdot 1_{\{\theta_1 \leq T\}} \cdot 1_{\{\tau^0 \geq \theta_1\} \cap \{\tau^1 < \theta_2\}} + Z_{\tau^2}^2 \cdot 1_{\{\theta_1, \theta_2 \leq T\}} \cdot 1_{\{\tau^0 \geq \theta_1\} \cap \{\tau^1 \geq \theta_2\}}. \end{aligned}$$

Therefore, we have

$$J_0 = \text{I} + \text{II} + \text{III} = \mathbb{E}\left[\int_{E^2} \int_{\Delta_2} Z_\tau(\theta_1, \theta_2, e_1, e_2) \cdot \alpha(\theta_1, \theta_2, e_1, e_2) d\theta_1 d\theta_2 \eta(de_1) \eta(de_2)\right] = \mathbb{E}[Z_T],$$

where the last equality follows from the expectation under the product probability measure. \square

Remark 3.4. In [17], a similar decomposition result for $\mathbb{E}[Z_T]$ is proved using the optional projection, where the Brownian motion and jump process can be dependent. In our case, since τ is a \mathbb{G} -stopping time, the optional projection cannot be used. (This step is the reason we assume the independence of the Brownian motion and jump process.)

4. DECOMPOSITION OF \mathbb{G} -CONTROLLER-STOPPER PROBLEMS

Theorem 4.2 and Proposition 4.4 are the main results for this section, which decompose the global \mathbb{G} -controller-stopper problems into a backward induction, where each step is a controller-stopper problem with respect to the Brownian filtration.

A control is a \mathbb{G} -predictable process $\pi \sim (\pi^0, \dots, \pi^n)$, where $\pi^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is valued in a set A^k in some Euclidian space, for $k = 0, \dots, n$. We denote by $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k; A^k)$ the set of elements in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ valued in A^k , $k = 0, \dots, n$. We require that all the \mathbb{G} -stopping times we consider here are valued in $[0, T]$, where $T \in (0, \infty]$ is a given constant. A trading strategy is a pair of a control and a \mathbb{G} -stopping time. We will use the notation $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ for the trading strategy if $\pi \sim (\pi^0, \dots, \pi^n)$ and $\tau \sim (\tau^0, \dots, \tau^n)$. A trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ is admissible, if for $k = 0, \dots, n$, $(\pi^k, \tau^k) \in \mathcal{A}^k \times \mathcal{T}^k$, where \mathcal{A}^k is some separable metric space of $\mathcal{P}(\Delta_k, E^k; A^k)$,

and \mathcal{T}^k is some set of finite \mathbb{G}^k -stopping times. By Proposition 2.4, we let \mathcal{T}^k be such that for any $\tau^k \in \mathcal{T}^k$, $\tau^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \leq T$ whenever $\theta_k \leq T$. Note that \mathcal{A}^k and \mathcal{T}^k may depend on each other in general. We denote the set of admissible trading strategies by $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$.

The following lemma will be used for the measurable selection issue later on.

Lemma 4.1. *For $k = 0, \dots, n$, define the metric on \mathcal{T}^k in the following way:*

$$\rho(\tau_1^k, \tau_2^k) := \mathbb{E} \left[\int_0^\infty e^{-t} |1_{\{\tau_1^k \leq t\}} - 1_{\{\tau_2^k \leq t\}}| dt \right], \quad \tau_1^k, \tau_2^k \in \mathcal{T}^k.$$

Then \mathcal{T}^k is a separable metric space.

Proof. Since for any \mathbb{G}^k -stopping time τ^k , $e^{-t} 1_{\{\tau^k \leq t\}}$ is a \mathbb{G}^k -adapted process in $L^1([0, \infty) \times \Omega)$, the conclusion follows from the separability of L^1 , see [18]. \square

Following [17], we describe the formulation of a stopped controlled state process as follows:

- Controlled state process between jumps:

$$(x, \pi^k) \in \mathbb{R}^d \times \mathcal{A}^k \mapsto X^{k,x,\pi^k} \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k), \quad k = 0, \dots, n,$$

such that

$$X_0^{0,x,\pi^0} = x, \quad X_{\theta_k}^{k,\beta,\pi^k}(\boldsymbol{\theta}_k, \mathbf{e}_k) = \beta, \quad \forall \beta \text{ } \mathcal{F}_{\theta_k}\text{-measurable.}$$

- Jumps of controlled state process: we have a collection of maps Γ^k on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A^{k-1} \times E$, for $k = 1, \dots, n$, such that

$$(t, \omega, x, a, e) \mapsto \Gamma^k(\omega, x, a, e) \text{ is } \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A^{k-1}) \otimes \mathcal{B}(E)\text{-measurable}$$

- Global controlled state process:

$$(x, \pi \sim (\pi^0, \dots, \pi^n)) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{G}} \mapsto X^{x,\pi} \in \mathcal{O}_{\mathbb{G}},$$

where

$$(4.1) \quad X_t^{x,\pi} = \bar{X}_t^0 1_{\{t < \zeta_1\}} + \sum_{k=1}^{n-1} \bar{X}_t^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) 1_{\{\zeta_k \leq t < \zeta_{k+1}\}} + \bar{X}_t^n(\boldsymbol{\zeta}_n, \boldsymbol{\ell}_n) 1_{\{\zeta_n \leq t\}},$$

with $(\bar{X}^0, \dots, \bar{X}^n) \in \mathcal{O}_{\mathbb{F}}(\Delta_0, E^0) \times \dots \times \mathcal{O}_{\mathbb{F}}(\Delta_n, E^n)$ with initial data

$$\begin{aligned} \bar{X}^0 &= X^{0,x,\pi^0}, \\ \bar{X}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) &= X^{k,\Gamma_{\theta_k}^k(\bar{X}_{\theta_k}^{k-1}, \pi_{\theta_k}^{k-1}, \mathbf{e}_k), \pi^k}(\boldsymbol{\theta}_k, \mathbf{e}_k), \quad k = 1, \dots, n. \end{aligned}$$

- Stopped global controlled state process: given a trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, let $X^{x,\pi}$ be the process from (4.1), then the stopped controlled state process is:

$$(4.2) \quad X_\tau^{x,\pi} = \bar{X}_\tau^0 1_{\{\tau < \zeta_1\}} + \sum_{k=1}^{n-1} \bar{X}_\tau^k(\boldsymbol{\zeta}_k, \boldsymbol{\ell}_k) 1_{\{\zeta_k \leq \tau < \zeta_{k+1}\}} + \bar{X}_\tau^n(\boldsymbol{\zeta}_n, \boldsymbol{\ell}_n) 1_{\{\zeta_n \leq \tau\}}.$$

Assume $U \sim (U^0, \dots, U^n)$ is a bounded (nonnegative, nonpositive) \mathbb{G} -optional process, which gives the terminal payoff U_t at time t . Consider the two types of the controller-stopper problems:

$$(4.3) \quad V^0(x) = \sup_{\tau \in \mathcal{T}_{\mathbb{G}}} \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[U_{\tau}(X_{\tau}^{x, \pi})], \quad x \in \mathbb{R}^d,$$

$$(4.4) \quad \mathfrak{V}^0(x) = \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \inf_{\tau \in \mathcal{T}_{\mathbb{G}}} \mathbb{E}[U_{\tau}(X_{\tau}^{x, \pi})], \quad x \in \mathbb{R}^d,$$

The following theorem provides a decomposition for calculating V^0 in (4.3). Its proof is similar to the proof of Theorem 4.1 in [17].

Theorem 4.2. *Define value functions $(\bar{V}^k)_{k=0}^n$ as*

$$(4.5) \quad \begin{aligned} \bar{V}^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n) &= \text{ess sup}_{\tau^n \in \mathcal{T}^n} \text{ess sup}_{\pi^n \in \mathcal{A}^n} \mathbb{E} \left[U_{\tau^n}^n(X_{\tau^n}^{n, x, \pi^n}, \boldsymbol{\theta}_n, \mathbf{e}_n) \cdot \alpha(\boldsymbol{\theta}_n, \mathbf{e}_n) | \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n, \\ \bar{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \text{ess sup}_{\tau^k \in \mathcal{T}^k} \text{ess sup}_{\pi^k \in \mathcal{A}^k} \mathbb{E} \left[U_{\tau^k}^k(X_{\tau^k}^{k, x, \pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1} \left(\Gamma_{\theta_k}^{k+1}(X_{\theta_{k+1}}^{k, x, \pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1} | \mathcal{F}_{\theta_k} \right], \\ &\quad (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k, \text{ for } k = 0, \dots, n-1. \text{ Then } V^0(x) = \bar{V}^0(x). \end{aligned}$$

Remark 4.3. In Equation (4.5), the first term $U_{\tau^k}^k(X_{\tau^k}^{k, x, \pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$ can be interpreted as the gain when there are no jumps between θ_k and τ^k , which is measured by the survival density $\alpha_{\tau^k}^k$. The second term

$$\int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1} \left(\Gamma_{\theta_k}^{k+1}(X_{\theta_{k+1}}^{k, x, \pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1}$$

can be understood as the gain when there is a jump at time θ_{k+1} between θ_k and τ^k .

Proof of Theorem 4.2. For $x \in \mathbb{R}^d$, $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, define

$$\begin{aligned} I^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n, \pi, \tau) &= \mathbb{E} \left[U_{\tau^n}^n(X_{\tau^n}^{n, x, \pi^n}, \boldsymbol{\theta}_n, \mathbf{e}_n) \cdot \alpha(\boldsymbol{\theta}_n, \mathbf{e}_n) | \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n, \\ I^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi, \tau) &= \mathbb{E} \left[U_{\tau^k}^k(X_{\tau^k}^{k, x, \pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E I^{k+1} \left(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k, x, \pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}, \pi, \tau \right) \eta(de_{k+1}) d\theta_{k+1} | \mathcal{F}_{\theta_k} \right], \end{aligned}$$

$(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$. Set $\bar{I}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = I^k(\bar{X}_{\theta_k}^k, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi, \tau)$, $k = 0, \dots, n$. From the decomposition (4.2), we know that $(\bar{I}^k)_{k=0}^n$ satisfy the backward induction formula:

$$\begin{aligned} \bar{I}^n(\boldsymbol{\theta}_n, \mathbf{e}_n) &= \mathbb{E} \left[U_{\tau^n}^n(\bar{X}_{\tau^n}^n, \boldsymbol{\theta}_n, \mathbf{e}_n) \cdot \alpha(\boldsymbol{\theta}_n, \mathbf{e}_n) | \mathcal{F}_{\theta_n} \right], \\ \bar{I}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) &= \mathbb{E} \left[U_{\tau^k}^k(\bar{X}_{\tau^k}^k, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \int_{\theta_k}^{\tau^k} \int_E \bar{I}^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}) \eta(de_{k+1}) d\theta_{k+1} | \mathcal{F}_{\theta_k} \right]. \end{aligned}$$

From Theorem 3.3 we have that

$$(4.6) \quad \bar{I}^0 = I^0 = \mathbb{E}[U_{\tau}(X_{\tau}^{x, \pi})].$$

Define the value function processes

$$(4.7) \quad V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) := \operatorname{ess\,sup}_{\tau \in \mathcal{A}_{\mathbb{G}}} \operatorname{ess\,sup}_{\pi \in \mathcal{T}_{\mathbb{G}}} I^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi, \tau),$$

for $k = 0, \dots, n$, $x \in \mathbb{R}^d$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$. Observe that V^0 defined in (4.7) is consistent with its definition in (4.3) from (4.6). Then it remains to show that $\bar{V}^k = V^k$ for $k = 0, \dots, n$. For $k = n$, since $I^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n, \pi, \tau)$ in fact only depends on (π^n, τ^n) , we immediately have $\bar{V}^n = V^n$. Now assume $\bar{V}^{k+1} = V^{k+1}$, for $0 \leq k \leq n-1$. Then for any $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$,

$$\begin{aligned} I^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi, \tau) &\leq \mathbb{E} \left[U_{\tau^k}^k(X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E V^{k+1} \left(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right] \\ &\leq \bar{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k), \end{aligned}$$

which implies that $V^k \leq \bar{V}^k$.

Conversely, given $x \in \mathbb{R}^d$ and $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, let us prove $V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) \geq \bar{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k)$. Fix $(\pi^k, \tau^k) \in \mathcal{A}^k \times \mathcal{T}^k$ and the associated controlled process X^{k,x,π^k} , from the definition of V^{k+1} , we have that for any $\omega \in \Omega$ and $\epsilon > 0$, there exists $(\pi^{\omega,\epsilon}, \tau^{\omega,\epsilon}) \in \mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, such that it is an $\epsilon e^{-\theta_{k+1}}$ -optimal trading strategy for $V^{k+1}(\cdot, \boldsymbol{\theta}_k, \mathbf{e}_k)$ at $(\omega, \Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}))$. By the separability of the set of admissible trading strategies from Lemma 4.1, one can use a measurable selection argument (e.g., see [19]) to find $(\pi^\epsilon, \tau^\epsilon) \sim (\pi^{\epsilon,k}, \tau^{\epsilon,k})_{k=0}^n$ in $\mathcal{A}_{\mathbb{G}} \times \mathcal{T}_{\mathbb{G}}$, such that $\pi_t^\epsilon(\omega) = \pi_t^{\omega,\epsilon}(\omega)$, $dt \otimes d\mathbb{P}$ -a.e. and $\tau^\epsilon(\omega) = \tau^{\omega,\epsilon}(\omega)$, a.s., and thus

$$\begin{aligned} V^{k+1}(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}) - \epsilon e^{-\theta_{k+1}} \\ \leq I^{k+1}(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}, \pi^\epsilon, \tau^\epsilon), \quad \text{a.s.} \end{aligned}$$

Consider the admissible trading strategy $(\tilde{\pi}^\epsilon, \tilde{\tau}^\epsilon)$ with the decomposition

$$\tilde{\pi}^\epsilon \sim (\pi^{\epsilon,0}, \dots, \pi^{\epsilon,k-1}, \pi^k, \pi^{\epsilon,k+1}, \dots, \pi^{\epsilon,n}) \quad \text{and} \quad \tilde{\tau}^\epsilon \sim (\tau^{\epsilon,0}, \dots, \tau^{\epsilon,k-1}, \tau^k, \tau^{\epsilon,k+1}, \dots, \tau^{\epsilon,n}).$$

Since $I^{k+1}(x, \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}, \pi, \tau)$ depends on $(\pi, \tau) \sim (\pi^j, \tau^j)_{j=0}^n$ only through their last components $(\pi^j, \tau^j)_{j=k+1}^n$, we have

$$\begin{aligned} V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) &\geq I^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \tilde{\pi}^\epsilon, \tilde{\tau}^\epsilon) \\ &= \mathbb{E} \left[U_{\tau^k}^k(X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E I^{k+1} \left(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}, \tilde{\pi}^\epsilon, \tilde{\tau}^\epsilon \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right] \\ &\geq \mathbb{E} \left[U_{\tau^k}^k(X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E \bar{V}^{k+1} \left(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right] - \epsilon, \end{aligned}$$

Therefore, $V^k \geq \bar{V}^k$, from which the claim of the theorem follows. \square

Now let us consider the value function \mathfrak{V}_0 in (4.4). We have the following result:

Proposition 4.4. *Define value functions $(\tilde{\mathfrak{V}}^k)_{k=0}^n$ as*

$$\begin{aligned}\tilde{\mathfrak{V}}^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n) &= \operatorname{ess\,sup}_{\pi^n \in \mathcal{A}^n} \operatorname{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E} \left[U_{\tau^n}^n (X_{\tau^n}^{n,x,\pi^n}, \boldsymbol{\theta}_n, \mathbf{e}_n) \cdot \alpha(\boldsymbol{\theta}_n, \mathbf{e}_n) \middle| \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n, \\ \tilde{\mathfrak{V}}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \operatorname{ess\,sup}_{\pi^k \in \mathcal{A}^k} \operatorname{ess\,inf}_{\tau^k \in \mathcal{T}^k} \mathbb{E} \left[U_{\tau^k}^k (X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E \tilde{\mathfrak{V}}^{k+1} \left(\Gamma_{\theta_k}^{k+1} (X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right], \\ &\quad (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k, \text{ for } k = 0, \dots, n-1. \text{ Then } \mathfrak{V}_0(x) = \tilde{\mathfrak{V}}^0(x).\end{aligned}$$

Proof. Given $\pi \sim (\pi^0, \dots, \pi^n)$ in $\mathcal{A}_{\mathbb{G}}$, define

$$\begin{aligned}\tilde{\mathfrak{V}}^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n, \pi) &= \operatorname{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E} \left[U_{\tau^n}^n (X_{\tau^n}^{n,x,\pi^n}, \boldsymbol{\theta}_n, \mathbf{e}_n) \cdot \alpha(\boldsymbol{\theta}_n, \mathbf{e}_n) \middle| \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n, \\ \tilde{\mathfrak{V}}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi) &= \operatorname{ess\,inf}_{\tau^k \in \mathcal{T}^k} \mathbb{E} \left[U_{\tau^k}^k (X_{\tau^k}^{k,x,\pi^k}, \boldsymbol{\theta}_k, \mathbf{e}_k) \cdot \alpha_{\tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E \tilde{\mathfrak{V}}^{k+1} \left(\Gamma_{\theta_k}^{k+1} (X_{\theta_{k+1}}^{k,x,\pi^k}, \pi_{\theta_{k+1}}^k, e_{k+1}), \boldsymbol{\theta}_k, \theta_{k+1}, \mathbf{e}_k, e_{k+1}, \pi \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right], \\ &\quad (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k, \text{ for } k = 0, \dots, n-1. \text{ From Theorem 4.2, we have}\end{aligned}$$

$$\tilde{\mathfrak{V}}^0(x, \pi) = \inf_{\tau \in \mathcal{T}_{\mathbb{G}}} \mathbb{E} U_{\tau} (X_{\tau}^{x,\pi}).$$

Define

$$\mathfrak{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) := \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{G}}} \tilde{\mathfrak{V}}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \pi), \quad (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k, \quad k = 0, \dots, n.$$

Then the definition for \mathfrak{V}^0 above is consistent with (4.4). Following the proof of Theorem 4.2 we can show $\mathfrak{V}^k = \tilde{\mathfrak{V}}^k$, $k = 0, \dots, n$. \square

5. APPLICATION TO INDIFFERENCE PRICING OF AMERICAN OPTIONS

In this section, we apply our decomposition method to indifference pricing of American options under multiple default risk. The main results are Theorem 5.4 and Theorem 5.8, which provide the RBSDE characterization of the indifference prices.

5.1. Market model. The model we will use here is similar to that in [9]. Let $T \in (0, \infty)$ be the finite time horizon. We assume in the market, there exists at most n default events. Let ζ_1, \dots, ζ_n and ℓ_1, \dots, ℓ_n represent the random default times and marks respectively, with α defined in (3.1) as the probability density. For any time t , if $\zeta_k \leq t < \zeta_{k+1}$, $k = 1, \dots, n-1$ ($t < \zeta_1$ for $k = 0$ and $t \geq \zeta_n$ for $k = n$), we say the underlying processes are in the k -default scenario.

We consider a portfolio of d -asset with a value process defined by a d -dimensional \mathbb{G} -optional process $S \sim (S^0, \dots, S^n)$ from (3.4), where $S^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ is valued in \mathbb{R}_+^d , representing the asset value in the k -default scenario, given the past default times $\boldsymbol{\zeta}_k = \boldsymbol{\theta}_k$ and the associated marks $\boldsymbol{\ell}_k = \mathbf{e}_k$, for $k = 0, \dots, n$. Suppose the dynamics of the indexed process S^k is given by

$$(5.1) \quad dS_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = S_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) * \left(b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) dt + \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) dW_t \right), \quad t \geq \theta_k,$$

where W is an m -dimensional (\mathbb{P}, \mathbb{F}) -Brownian motion, $m \geq d$, b^k and σ^k are indexed processes in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued respectively in \mathbb{R}^d and $\mathbb{R}^{d \times m}$. Here, for $x = (x_1, \dots, x_d)' \in \mathbb{R}^d$, and $y = (y_1, \dots, y_d)' \in \mathbb{R}^{d \times q}$, the expression $x * y$ denotes the vector $(x_1 y_1, \dots, x_d y_d)' \in \mathbb{R}^{d \times q}$. Equation (5.1) can be viewed as an asset model with change of regimes after default events, with coefficient b^k, σ^k depending on the past default information. We make the usual no-arbitrage assumption that there exists an indexed risk premium process $\lambda^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, such that for all $(\theta_k, \mathbf{e}_k) \in \Delta_k \times E^k$,

$$(5.2) \quad \sigma_t^k(\theta_k, \mathbf{e}_k) \lambda_t^k(\theta_k, \mathbf{e}_k) = b_t^k(\theta_k, \mathbf{e}_k), \quad t \geq 0.$$

Moreover, each default time θ_k may induce a jump in the asset portfolio, which will be formalized by considering a family of indexed processes $\gamma^k \in \mathcal{P}(\Delta_k, E^k, E)$, valued in $[-1, \infty)$, for $k = 0, \dots, n-1$. For $(\theta_k, \mathbf{e}_k) \in \Delta_k \times E^k$ and $e_{k+1} \in E$, $\gamma_{\theta_{k+1}}^k(\theta_k, \mathbf{e}_k, e_{k+1})$ represents the relative vector jump size on the d assets at time $t = \theta_{k+1} \geq \theta_k$ with a mark e_{k+1} , given the past default events $(\zeta_k, \ell_k) = (\theta_k, \mathbf{e}_k)$. In other words, we have:

$$(5.3) \quad S_{\theta_{k+1}}^{k+1}(\theta_{k+1}, \mathbf{e}_{k+1}) = S_{\theta_{k+1}}^k(\theta_k, \mathbf{e}_k) * \left(\mathbf{1}_d + \gamma_{\theta_{k+1}}^k(\theta_k, \mathbf{e}_k, e_{k+1}) \right),$$

where $\mathbf{1}_d$ is the vector in \mathbb{R}^d with all components equal to 1.

Remark 5.1. It is possible that after default times, some assets may not be traded any more. Now suppose that after k defaults, there are \bar{d} assets still tradable, where $0 \leq \bar{d} \leq d$. Then without loss of generality, we may assume $b^k(\theta_k, \mathbf{e}_k) = (\bar{b}(\theta_k, \mathbf{e}_k) \ 0)$, $\sigma^k(\theta_k, \mathbf{e}_k) = (\bar{\sigma}^k(\theta_k, \mathbf{e}_k) \ 0)$, $\gamma^k(\theta_k, \mathbf{e}_k, e) = (\bar{\gamma}^k(\theta_k, \mathbf{e}_k, e) \ 0)$, where $\bar{b}(\theta_k, \mathbf{e}_k)$, $\bar{\sigma}^k(\theta_k, \mathbf{e}_k)$, $\bar{\gamma}^k(\theta_k, \mathbf{e}_k, e)$ are \mathbb{F} -predictable processes valued respectively in $\mathbb{R}^{\bar{d}}$, $\mathbb{R}^{\bar{d} \times m}$, $\mathbb{R}^{\bar{d}}$. In this case, we shall also assume that the volatility matrix $\bar{\sigma}^k(\theta_k, \mathbf{e}_k)$ is of full rank. we can then define the risk premium

$$\lambda^k(\theta_k, \mathbf{e}_k) = \bar{\sigma}^k(\theta_k, \mathbf{e}_k)' \left(\bar{\sigma}^k(\theta_k, \mathbf{e}_k) \bar{\sigma}^k(\theta_k, \mathbf{e}_k)' \right)^{-1} \bar{b}^k(\theta_k, \mathbf{e}_k),$$

which satisfies (5.2).

An American option of maturity T is modeled by a \mathbb{G} -optional process $R \sim (R^0, \dots, R^n)$ from (3.3), where $R_t^k(\theta_k, \mathbf{e}_k)$ is continuous with respect to t , and represents the payoff if the option is exercised at time $t \in [\theta_k, T]$ in the k -default scenario, given the past default events $(\zeta_k, \ell_k) = (\theta_k, \mathbf{e}_k)$, for $k = 0, \dots, n$.

A control in the d -asset portfolio is a \mathbb{G} -predictable process $\pi \sim (\pi^0, \dots, \pi^n)$, where $\pi^k(\theta_k, \mathbf{e}_k) \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ is valued in a closed set A^k of \mathbb{R}^d containing the zero element, and represents the amount invested continuously in the d assets in the k -default scenario, given the past default information $(\zeta_k, \ell_k) = (\theta_k, \mathbf{e}_k)$. An exercise time is a \mathbb{G} -stopping time $\tau \sim (\tau^0, \dots, \tau^n)$ satisfying $\tau \leq T$, with the decomposition from Proposition 2.4. A trading strategy is a pair of a control and an exercise time.

For a trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$, we have the corresponding wealth process $\mathfrak{X} \sim (\mathfrak{X}^0, \dots, \mathfrak{X}^n)$, where $\mathfrak{X}^k(\theta_k, \mathbf{e}_k) \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, representing the wealth controlled by $\pi^k(\theta_k, \mathbf{e}_k)$ in the price process $S^k(\theta_k, \mathbf{e}_k)$, given the past default events $(\zeta_k, \ell_k) = (\theta_k, \mathbf{e}_k)$. From (5.1) we have

$$d\mathfrak{X}_t^k(\theta_k, \mathbf{e}_k) = \pi_t^k(\theta_k, \mathbf{e}_k)' \left(b_t^k(\theta_k, \mathbf{e}_k) dt + \sigma^k(\theta_k, \mathbf{e}_k) dW_t \right), \quad t \geq \theta_k.$$

Moreover, each default time induces a jump in the asset price process, and then also on the wealth process. From (5.3), we have

$$(5.4) \quad \mathfrak{X}_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}) = \mathfrak{X}_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e_{k+1}).$$

5.2. Indifference price. Let U be an exponential utility with risk aversion coefficient $p > 0$:

$$U(x) = -\exp(-px), \quad x \in \mathbb{R},$$

which describes an investor's preference. We will consider two cases. The first case is that the investor can trade the d -assets portfolio following control π , associated to a wealth process $\mathfrak{X} = \mathfrak{X}^{x, \pi}$ with initial capital $\mathfrak{X}_{0-} = x$. Besides, she holds an American option and can choose to exercise it at any time τ , $\tau \leq T$, to get payoff R_τ . So the maximum utility she can get (or as close as she want, if not attainable) is:

$$(5.5) \quad V^0(x) = \sup_{\tau} \sup_{\pi} \mathbb{E} [U(\mathfrak{X}_{\tau}^{x, \pi} + R_{\tau})].$$

Then the maximum price \bar{c} the investor is willing to pay for the American option should satisfy:

$$U(x + \bar{c}) = V^0(x).$$

In this case, we call \bar{c} the indifference buying price of the American option.

The second case is that the investor trades the d -asset portfolio following control π , while shorting an American option. So she has to deliver the payoff R_τ at some exercise time τ , which is chosen by the holder of the option. By considering the worst scenario, the maximum utility she can get (or as close as she want) is:

$$(5.6) \quad \mathfrak{V}^0(x) = \sup_{\pi} \inf_{\tau} \mathbb{E} [U(\mathfrak{X}_{\tau}^{x, \pi} - R_{\tau})].$$

In this case, we call \underline{c} the indifference selling price of the American option, if

$$U(x - \underline{c}) = \mathfrak{V}^0(x).$$

5.3. Indifference buying price. In this sub-section, we will focus on the problem (5.5). Theorem 5.4 is the main result for this sub-section.

Definition 5.2. (Admissible trading strategy) A trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ is admissible, if for any $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, under the control π^k ,

- (a) $\int_{\theta_k}^{\tau^k} |\pi_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)| dt + \int_{\theta_k}^{\tau^k} |\pi_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 dt < \infty$, a.s., $k = 0, \dots, n$,
- (b) the family $\left\{ U(\mathfrak{X}_{\tau \wedge \tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) : \tau \text{ is any } \mathbb{F} - \text{stopping time valued in } [\theta_k, T] \right\}$ is uniformly integrable, i.e., $U(\mathfrak{X}_{\cdot \wedge \tau^k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k))$ is of class (D), for $k = 0, \dots, n$,
- (c) $\mathbb{E} \left[\int_{\theta_k}^{\tau^k} \int_E (-U) \left(\mathfrak{X}_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \pi_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \gamma_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) \right) \eta(de) ds \right] < \infty$, for $k = 0, \dots, n-1$.

The notation $\mathcal{A}_{\mathbb{G}}$, $\mathcal{T}_{\mathbb{G}}$, \mathcal{A}^k and \mathcal{T}^k from Section 5 are now specified by the above definition. From Theorem 4.2, V^0 in (5.5) can be calculated by the following backward induction:

$$(5.7) \quad V^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n) = \operatorname{ess\,sup}_{\tau^n \in \mathcal{T}^n} \operatorname{ess\,sup}_{\pi^n \in \mathcal{A}^n} \mathbb{E} \left[U(\mathfrak{X}_{\tau^n}^{n,x} + H_{\tau^n}^n) | \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n,$$

$$(5.8) \quad V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = \operatorname{ess\,sup}_{\tau^k \in \mathcal{T}^k} \operatorname{ess\,sup}_{\pi^k \in \mathcal{A}^k} \mathbb{E} \left[U(\mathfrak{X}_{\tau^k}^{k,x} + H_{\tau^k}^k) \right. \\ \left. + \int_{\theta_k}^{\tau^k} \int_E V^{k+1}(\mathfrak{X}_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \cdot \gamma_{\theta_{k+1}}^k(e_{k+1}), \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}) \eta(de_{k+1}) d\theta_{k+1} | \mathcal{F}_{\theta_k} \right],$$

$(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$, where

$$H^k := R^k - \frac{1}{p} \ln \alpha^k,$$

in which α^k is given by (3.2).

5.3.1. *Backward recursive system of RBSDEs.* Following [7], we expect the value function to be of the following form:

$$(5.9) \quad V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = U \left(x + Y_{\theta_k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right),$$

where $Y^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$ is an \mathbb{F} -adapted process, satisfying the RBSDE $\mathbf{eq}(H^k(\boldsymbol{\theta}_k, \mathbf{e}_k), f^k)_{\theta_k \leq t \leq T}$, with f^k defined as

$$(5.10) \quad f^k(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) = \inf_{\pi \in A^k} g^k(\pi, t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k),$$

where

$$\begin{aligned} g^k(\pi, t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \frac{p}{2} \left| z - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \right|^2 - b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \\ &\quad + \frac{1}{p} U(-y) \int_E U \left(\pi \cdot \gamma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e) \right) \eta(de) \\ &= -\lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \cdot z - \frac{1}{2p} |\lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 + \frac{p}{2} \left| z + \frac{1}{p} \lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \right|^2 \\ &\quad + \frac{1}{p} U(-y) \int_E U \left(\pi \cdot \gamma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e) \right) \eta(de), \end{aligned}$$

for $k = 0, \dots, n-1$, and

$$\begin{aligned} g^n(\pi, t, y, z, \boldsymbol{\theta}_n, \mathbf{e}_n) &= \frac{p}{2} \left| z - \sigma_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)' \pi \right|^2 - b_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)' \pi \\ &= -\lambda_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n) \cdot z - \frac{1}{2p} |\lambda_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)|^2 + \frac{p}{2} \left| z + \frac{1}{p} \lambda_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n) - \sigma_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)' \pi \right|^2. \end{aligned}$$

In the next two subsections, we will show that: (a) The backward recursive system of RBSDEs admits a solution; (b) The solution characterizes the values of (V^k) , i.e., (5.9) holds.

5.3.2. *Existence to the recursive system of RBSDEs.* We make the following boundedness assumptions **(HB)**:

- (i) The risk premium is bounded uniformly with respect to its indices: there exists a constant $C > 0$, such that for any $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, $t \in [\theta_k, T]$,

$$|\lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)| \leq C, \quad \text{a.s.}$$

- (ii) The indexed random variables $(H_t^k)_k$ are bounded uniformly in time and their indices: there exists a constant $C > 0$ such that for any $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, $t \in [\theta_k, T]$,

$$|H_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)| \leq C, \quad \text{a.s.}$$

Theorem 5.3. *Under **(HB)**, there exists a solution $(Y^k, Z^k, K^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to the recursive system of indexed RBSDEs $\mathbf{eq}(H^k(\boldsymbol{\theta}_k, \mathbf{e}_k), f^k)_{\theta_k \leq t \leq T}$, $k = 0, \dots, n$.*

Proof. We prove the result by a backward induction on $k = 0, \dots, n$. The positive constant C may vary from line to line, but is always independent of $(t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k)$. We will often omit the dependence of $(t, \omega, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k)$ in related functions.

(a) For $k = n$. Under **(HB)**, $|f^n| \leq C(|z|^2 + 1)$. By Theorem 1 in [15], there exists a solution $(Y^n(\boldsymbol{\theta}_n, \mathbf{e}_n), Z^n(\boldsymbol{\theta}_n, \mathbf{e}_n), K^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) \in \mathcal{S}_c^\infty[\theta_n, T] \times \mathbf{L}_W^2[\theta_n, T] \times \mathbf{A}[\theta_n, T]$ for $\mathbf{eq}(H^n, f^n)_{\theta_n \leq t \leq T}$, satisfying $|Y^n| \leq C$. Moreover, the measurability of (Y^n, Z^n) with respect to $(\boldsymbol{\theta}_n, \mathbf{e}_n)$ follows from the measurability of H^n and f^n (see Appendix C in [14] and use the fact that the solution to the RBSDE can be eventually approximated by the solutions to BSDEs). Therefore, $(Y^n, Z^n, K^n) \in \mathcal{S}_c^\infty(\Delta_n(T), E^n) \times \mathbf{L}_W^2(\Delta_n(T), E^n) \times \mathbf{A}(\Delta_n(T), E^n)$.

(b) For $k \in \{0, 1, \dots, n-1\}$. Assume there exists $(Y^{k+1}, Z^{k+1}, K^{k+1}) \in \mathcal{S}_c^\infty(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{L}_W^2(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{A}(\Delta_{k+1}(T), E^{k+1})$ satisfying $\mathbf{eq}(H^{k+1}, f^{k+1})$. Since $Y^{k+1} \in \mathcal{P}_{\mathbb{F}}(\Delta_{k+1}, E^{k+1})$, the generator in (5.10) is well defined. In order to overcome the technical difficulties coming from the exponential term in $U(-y)$, we first consider the truncated generator

$$f^{k,N}(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) = \inf_{\pi \in A^k} g^k(\pi, t, N \wedge y, z, \boldsymbol{\theta}_k, \mathbf{e}_k).$$

Then there exists a positive constant C_N independent of $(\boldsymbol{\theta}_k, \mathbf{e}_k)$, such that $|f^{k,N}| \leq C_N(1 + z^2)$. Applying Theorem 1 in [15], there exists a solution $(Y^{k,N}, Z^{k,N}, K^{k,N}) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to $\mathbf{eq}(H^k, f^{k,N})$.

Now we will show that $Y^{k,N}$ has a uniform upper bound. Consider the generator

$$\bar{f}^k(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) := -\lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \cdot z - \frac{1}{2p} |\lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2,$$

which satisfies the Lipschitz condition in (y, z) , uniformly in (t, ω) . Then by Theorem 5.2 in [6], there exists a unique solution $(\bar{Y}^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \bar{Z}^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \bar{K}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) \in \mathcal{S}_c^\infty[\theta_k, T] \times \mathbf{L}_W^2[\theta_k, T] \times \mathbf{A}[\theta_k, T]$ satisfying $|\bar{Y}^k| \leq C$ (see Theorem 1 in [15] for the boundedness). Applying Lemma 2.1 (comparison) in [15], we get $Y^{k,N} \leq \bar{Y}^k$. Hence, $Y^{k,N}$ has a uniform upper bound independent of N and $(\boldsymbol{\theta}_k, \mathbf{e}_k)$. Therefore, for N large enough, we can remove “ N ” in the truncated generator $f^{k,N}$, i.e., $(Y^{k,N}, Z^{k,N}, K^{k,N})$ solves $\mathbf{eq}(H^k, f^k)$ for large enough N . \square

5.3.3. RBSDE characterization by verification theorem.

Theorem 5.4. *The value functions $(V^k)_{k=0}^n$, defined in (5.7) and (5.8), are given by*

$$(5.11) \quad V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = U(x + Y_{\boldsymbol{\theta}_k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)),$$

for $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, where $(Y^k, Z^k, K^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ is a solution of the RBSDE system $\mathbf{eq}(H^k, f^k)$, $k = 0, \dots, n$. Moreover, there exists an optimal trading strategy $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ described by:

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \arg \min_{\pi \in A^k} g^k(\pi, t, Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \boldsymbol{\theta}_k, \mathbf{e}_k),$$

for $t \in [\theta_k, T]$, and

$$(5.12) \quad \hat{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) := \inf \left\{ t \geq \theta_k : Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = H_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right\},$$

for $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, a.s., $k = 0, \dots, n$.

Proof. Step 1: We will show

$$(5.13) \quad U(x + Y_{\boldsymbol{\theta}_k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) \geq V^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k), \quad k = 0, \dots, n.$$

Let $(Y^k, Z^k, K^k) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ be a solution of the RBSDE system. For $(\nu^k, \tau^k) \in \mathcal{A}^k \times \mathcal{T}^k$, $x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$ and $t \geq \theta_k$, and define

$$\begin{aligned} \xi_t^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k) &:= U(\mathfrak{X}_t^{k,x} + Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) + \int_{\theta_k}^t \int_E U(\mathfrak{X}_r^{k,x} + \nu_r^k \cdot \gamma_r^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) \\ &\quad + Y_r^{k+1}(\boldsymbol{\theta}_k, r, \mathbf{e}_k, e)) \eta(dr, de), \quad k = 0, \dots, n-1, \\ \xi_t^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n, \nu^n) &:= U(\mathfrak{X}_t^{n,x} + Y_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)). \end{aligned}$$

Applying Itô's formula, we get for $k = 0, \dots, n$,

$$\begin{aligned} \xi_t^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k) &= pU(\mathfrak{X}_t^{k,x} + Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) \left[(-f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + g^k(\nu_t^k, t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k)) dt + dK_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + (Z_t^k - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \nu_t^k) \cdot dW_t \right], \end{aligned}$$

$f^k(\cdot) = \inf_{\pi \in A^k} g^k(\pi, \cdot)$ implies $\left\{ \xi_s^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k) \right\}_{\theta_k \leq s \leq T}$ is a local super-martingale, for $k = 0, \dots, n$. Since Y^k and Y^{k+1} are essentially bounded, and $\xi_{t \wedge \tau^k \wedge \rho_m}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k)$ is uniformly integrable, by considering a localizing sequence of stopping times, we can show $\left\{ \xi_{t \wedge \tau^k}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k) \right\}_{\theta_k \leq t \leq T}$ is a super-martingale. Consider when $k = n$. Since $Y^n \geq H^n$, we have

$$(5.14) \quad U(x + Y_{\boldsymbol{\theta}_n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) \geq \mathbb{E}[U(\mathfrak{X}_{\tau^n}^{n,x} + H_{\tau^n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) | \mathcal{F}_{\theta_n}].$$

Therefore, (5.13) holds for $k = n$. Similarly, it holds for $k = 0, \dots, n-1$.

Step 2: $\int_{\theta_k}^\cdot Z_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \cdot dW_s$ is a BMO-martingale. Apply Itô's formula to $\exp(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k))$

with $q > p$ and any \mathbb{F} -stopping time τ valued in $[\theta_k, T]$,

$$\begin{aligned} & \frac{1}{2}q(q-p)\mathbb{E}\left[\int_{\tau}^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) |Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 dt \middle| \mathcal{F}_{\tau}\right] \\ &= q\mathbb{E}\left[\int_{\tau}^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) \left(f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k) - \frac{p}{2}|Z_t^k|^2\right) dt \middle| \mathcal{F}_{\tau}\right] \\ &+ \mathbb{E}\left[\exp\left(-qY_T^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) - \exp\left(-qY_{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) \middle| \mathcal{F}_{\tau}\right] \\ &- q\mathbb{E}\left[\int_{\tau}^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) dK_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \middle| \mathcal{F}_{\tau}\right]. \end{aligned}$$

Since $|f^k(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k)| \leq \frac{p}{2}|z|^2 - CU(-y)$, $dK^k \geq 0$ and Y^k is bounded, we have

$$\begin{aligned} & \frac{1}{2}q(q-p)\mathbb{E}\left[\int_{\tau}^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) |Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 dt \middle| \mathcal{F}_{\tau}\right] \\ & \leq qC\mathbb{E}\left[\int_{\tau}^T \exp\left(-qY_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\right) dt \middle| \mathcal{F}_{\tau}\right] + C. \end{aligned}$$

By choosing q large enough, we have

$$\mathbb{E}\left[\int_{\tau}^T |Z_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 ds \middle| \mathcal{F}_{\tau}\right] \leq C,$$

which implies $\int_{\theta_k}^{\cdot} Z_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \cdot dW_s$ is a BMO-martingale.

Step 3: Admissibility of $(\hat{\pi}^k, \hat{\tau}^k)$. For $k = 0, \dots, n$, define function \hat{g}^k by

$$\hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k) = g^k(\pi, t, Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \boldsymbol{\theta}_k, \mathbf{e}_k).$$

We can show that the map $(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k) \rightarrow \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E_k)$ -measurable. Now for $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, if either $\sigma^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = 0$ or $\gamma^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) = 0$, then the continuous function $\pi \rightarrow \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k)$ attains trivially its infimum of \hat{g}^k when $\pi = 0$. Otherwise, $\sigma^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$ and $\gamma^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e)$ are in the form $\sigma^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = (\bar{\sigma}^k(\boldsymbol{\theta}_k, \mathbf{e}_k), 0)$, $\gamma^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = (\bar{\gamma}(\boldsymbol{\theta}_k, \mathbf{e}_k), 0)$ for some full rank matrix $\bar{\sigma}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$. In this case, we let $(\bar{\pi}, 0) = (\sigma^k)' \cdot \pi$, then we get

$$\begin{aligned} \bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k) &:= \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k) = \frac{p}{2} \left| Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \frac{1}{p} \lambda_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) - \bar{\pi} \right|^2 \\ &+ \frac{1}{p} U(-Y_t^k) \int_E U\left((\bar{\sigma}^k)'^{-1} \cdot \bar{\pi} \cdot \bar{\gamma}_t^k(e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e)\right) \eta(de), \end{aligned}$$

for $k = 0, \dots, n-1$, and

$$\bar{g}^n(\bar{\pi}, t, \omega, \boldsymbol{\theta}_n, \mathbf{e}_n) := \hat{g}^n(\pi, t, \omega, \boldsymbol{\theta}_n, \mathbf{e}_n) = \frac{p}{2} \left| Z_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n) + \frac{1}{p} \lambda_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n) - \bar{\pi} \right|^2.$$

Since

$$\bar{g}^k(0, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k) < \liminf_{|\bar{\pi}| \rightarrow \infty} \bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k),$$

the continuous function $\bar{\pi} \rightarrow \bar{g}^k(\bar{\pi}, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k)$ attains its infimum over the closed set $(\sigma_t^k)' A^k$, and thus the function $\pi \rightarrow \hat{g}^k(\pi, t, \omega, \boldsymbol{\theta}_k, \mathbf{e}_k)$ attains its infimum over $A^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$. For $k = 0, \dots, n$, using

a measurable selection argument (see [19]), one can show that there exists $\hat{\pi}^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, such that

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \arg \min_{\pi \in A^k(\boldsymbol{\theta}_k, \mathbf{e}_k)} \hat{g}^k(\pi, t, \boldsymbol{\theta}_k, \mathbf{e}_k), \quad \theta_k \leq t \leq T, \quad \text{a.s.}$$

Consider $\hat{\tau}^k$ defined in (5.12). For $k = 0, \dots, n$, define $\hat{\tau}^k(\zeta_k, \boldsymbol{\ell}_k)$ as

$$\hat{\tau}^k := \left(\inf \{t \geq \zeta_k : Y^k(\zeta_k, \boldsymbol{\ell}_k) = H^k(\zeta_k, \boldsymbol{\ell}_k)\} \wedge T \right) \cdot 1_{\{\zeta_k \leq T\}} + \zeta_k \cdot 1_{\{\zeta_k > T\}}.$$

We can show that $\hat{\tau}^k(\zeta_k, \boldsymbol{\ell}_k)$ is a \mathbb{G}^k stopping time satisfying $\hat{\tau}^k(\zeta_k, \boldsymbol{\ell}_k) \geq \zeta_k$ and $\{\hat{\tau}^k(\zeta_k, \boldsymbol{\ell}_k) \leq T\} = \{\zeta_k \leq T\}$. And given $(\zeta_k, \boldsymbol{\ell}_k) = (\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, $\hat{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = \hat{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$. Now we will show that $(\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ is admissible in the sense of Definition 5.2.

(a) Since $\hat{g}^k(\hat{\pi}_t^k, t, \boldsymbol{\theta}_k, \mathbf{e}_k) \leq \hat{g}^k(0, t, \boldsymbol{\theta}_k, \mathbf{e}_k)$, there exists a constant $C > 0$, such that

$$|\sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \hat{\pi}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)| \leq C(1 + |Z_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|), \quad \theta_k \leq t \leq T, \quad \text{a.s.},$$

for all $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, $k = 0, \dots, n$. Since $Z^k \in \mathbf{L}_W^2(\Delta_k, E^k)$ and because of (HB)(i), $(\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ satisfies condition (a) in Definition 5.2.

(b) Denote by $\hat{\mathcal{X}}^{k,x}$ the wealth process controlled by $\hat{\pi}^k$, starting from x at time θ_k . We have

$$f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k) = g^k(\hat{\pi}_t^k, t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k),$$

for $k = 0, \dots, n$. Then for $\theta_k \leq t \leq T$,

$$U(\hat{\mathcal{X}}_t^{k,x} + Y_t^k) = U(x + Y_{\theta_k}^k) \mathcal{E}_t^k \left(p(Z^k - (\sigma^k)' \hat{\pi}^k) \right) R_t^k,$$

where

$$\mathcal{E}_t^k \left(p(Z^k - (\sigma^k)' \hat{\pi}^k) \right) = \exp \left(p \int_{\theta_k}^t (Z_s^k - (\sigma_s^k)' \hat{\pi}_s^k) \cdot dW_s - \frac{p^2}{2} \int_{\theta_k}^t |Z_s^k - (\sigma_s^k)' \hat{\pi}_s^k|^2 ds \right),$$

for $k = 0, \dots, n$, and

$$R_t^k = \exp \left(pK_t^k - \int_{\theta_k}^t U(-Y_s^k) \int_E U \left(\hat{\pi}_t^k \cdot \gamma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e) \right) \eta(de) ds \right),$$

for $k = 0, \dots, n-1$ and $R_t^n = \exp(pK_t^n)$. From Step 2, $\int_{\theta_k}^\cdot p(Z^k - (\sigma^k)' \hat{\pi}^k) \cdot dW$ is a BMO-martingale and hence $\mathcal{E}_{\cdot \wedge \hat{\tau}^k}^k(p(Z^k - (\sigma^k)' \hat{\pi}^k))$ is of class (D). Moreover, since U is nonpositive and $K_t^k = 0$ when $t \leq \hat{\tau}^k$, we have $|R_{\cdot \wedge \hat{\tau}^k}| \leq 1$, and thus $U(\hat{\mathcal{X}}_{t \wedge \hat{\tau}^k}^{k,x} + Y_{t \wedge \hat{\tau}^k}^k)$ is of class (D). So is $U(\hat{\mathcal{X}}_{\cdot \wedge \hat{\tau}^k}^{k,x})$ since Y^k is essentially bounded.

(c) Because $dK_t^k = 0$ when $t \leq \hat{\tau}^k$, the process $\xi_{\cdot \wedge \hat{\tau}^k}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, e)$ defined in Step 1 under control $\hat{\pi}^k$ is a local martingale. By considering a localizing \mathbb{F} -stopping time sequence $(\rho_m)_m$ valued in $[\theta_k, T]$, we obtain:

$$\begin{aligned} & \mathbb{E} \left[\int_{\theta_k}^{\hat{\tau}^k \wedge \rho_m} \int_E (-U) \left(\hat{\mathcal{X}}_t^{k,x} + \hat{\pi}_t^k \cdot \gamma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) + Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e) \right) \eta(de) dt \right] \\ &= \mathbb{E} \left[U(\hat{\mathcal{X}}_{\hat{\tau}^k \wedge \rho_m}^{k,x} + Y_{\hat{\tau}^k \wedge \rho_m}^k) - U(x + Y_{\theta_k}^k) \right] \leq \mathbb{E} \left[-U(x + Y_{\theta_k}^k) \right], \end{aligned}$$

By Fatou's lemma, we get Condition (c) in Definition 5.2 holds.

Step 4: We will show (5.11) holds and $(\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ is an optimal trading strategy. Consider

when $k = n$. By the admissibility of $(\hat{\pi}^n, \hat{\tau}^n)$, the local martingale $\xi_{t \wedge \hat{\tau}^n}$ under the control $\hat{\pi}^n$ is a martingale. Thus,

$$U(x + Y_{\theta_n}^n) = \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} + H_{\hat{\tau}^n}^n) \middle| \mathcal{F}_{\theta_n} \right].$$

Along with (5.14) this results in

$$\begin{aligned} V^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n) &= \operatorname{ess\,sup}_{\tau^n \in \mathcal{T}^n} \operatorname{ess\,sup}_{\pi^n \in \mathcal{A}^n} \mathbb{E} \left[U(\mathfrak{X}_{\tau^n}^{n,x} + H_{\tau^n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) \middle| \mathcal{F}_{\theta_n} \right] \leq U(x + Y_{\theta_n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) \\ &= \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} + H_{\hat{\tau}^n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) \middle| \mathcal{F}_{\theta_n} \right] \leq V^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n), \end{aligned}$$

which implies (5.11) for $k = n$ and the optimality of $(\hat{\pi}^n, \hat{\tau}^n)$. We can show (5.11) and the optimality of $(\hat{\pi}^k, \hat{\tau}^k)$ for $k = 0, \dots, n-1$, similarly using (5.8). \square

5.4. Indifference selling price. In this sub-section, we consider the problem (5.6), and Theorem 5.8 is the main result.

Definition 5.5. (Admissible trading strategy) A trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$ is admissible, if for any $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, under the control π^k ,

- (a) $\int_{\theta_k}^T |\pi_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)| dt + \int_{\theta_k}^T |\pi_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)|^2 dt < \infty$, a.s., $k = 0, \dots, n$,
- (b) the family $\left\{ U(\mathfrak{X}_{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) : \tau \text{ is any } \mathbb{F} - \text{stopping time valued in } [\theta_k, T] \right\}$ is uniformly integrable, i.e., $U(\mathfrak{X}^k(\boldsymbol{\theta}_k, \mathbf{e}_k))$ is of class (D), for $k = 0, \dots, n$,
- (c) $\mathbb{E} \left[\int_{\theta_k}^T \int_E (-U) \left(\mathfrak{X}_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \pi_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \gamma_s^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) \right) \eta(de) ds \right] < \infty$, for $k = 0, \dots, n-1$.

Remark 5.6. Unlike in Definition 4.1, the admissible trading strategy here is in fact independent of stopping times. This is because the investor cannot choose when to stop.

5.4.1. Backward recursive system of RBSDEs. We decompose \mathfrak{V}^0 in (5.6) into a backward induction as before:

$$(5.15) \quad \mathfrak{V}^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n) = \operatorname{ess\,sup}_{\pi^n \in \mathcal{A}^n} \operatorname{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E} \left[U(\mathfrak{X}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) \middle| \mathcal{F}_{\theta_n} \right], \quad (\boldsymbol{\theta}_n, \mathbf{e}_n) \in \Delta_n(T) \times E^n,$$

$$(5.16) \quad \begin{aligned} \mathfrak{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \operatorname{ess\,sup}_{\pi^k \in \mathcal{A}^k} \operatorname{ess\,inf}_{\tau^k \in \mathcal{T}^k} \mathbb{E} \left[U(\mathfrak{X}_{\tau^k}^{k,x} - \mathcal{H}_{\tau^k}^k) \right. \\ &\quad \left. + \int_{\theta_k}^{\tau^k} \int_E \mathfrak{V}^{k+1} \left(\mathfrak{X}_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \cdot \gamma_{\theta_{k+1}}^k(e_{k+1}), \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1} \right) \eta(de_{k+1}) d\theta_{k+1} \middle| \mathcal{F}_{\theta_k} \right], \end{aligned}$$

$(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, for $k = 0, \dots, n-1$, where

$$\mathcal{H}^k = R^k + \frac{1}{p} \ln \alpha^k, \quad k = 0, \dots, n.$$

Consider

$$\mathfrak{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = U(x - \mathcal{Y}_{\theta_k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)), \quad k = 0, \dots, n,$$

where $\{\mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)\}_{k=0}^n$ satisfies the RBSDE $\mathbf{EQ}(\mathcal{H}^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \mathfrak{f}^k)_{\theta_k \leq t \leq T}$, with \mathfrak{f}^k defined as

$$\mathfrak{f}^k(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) = \inf_{\pi \in \mathcal{A}^k} \mathfrak{g}^k(\pi, t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k),$$

where

$$\begin{aligned} \mathbf{g}^k(\pi, t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \frac{p}{2} \left| z - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \right|^2 - b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \\ &\quad + \frac{1}{p} U(y) \int_E U \left(\pi \cdot \gamma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) - \mathcal{Y}_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e) \right) \eta(de) \end{aligned}$$

for $k = 0, \dots, n-1$, and

$$\mathbf{g}^n(\pi, t, y, z, \boldsymbol{\theta}_n, \mathbf{e}_n) = \frac{p}{2} \left| z - \sigma_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)' \pi \right|^2 - b_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)' \pi.$$

5.4.2. Existence to the recursive system of RBSDEs. We will make the same boundedness assumption as **(HB)** in Section 5.3.2 except that we will replace H^k with \mathcal{H}^k . Let us denote this assumption by **(HB')**.

Theorem 5.7. *Under **(HB')**, there exists a solution $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to the recursive system of indexed RBSDEs $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^k)$, $k = 0, \dots, n$.*

Proof. We prove the result by a backward induction on $k = 0, \dots, n$

For $k = n$. Using the same argument as in the proof of Theorem 5.3, we can show that there exists a solution $(\mathcal{Y}^n, \mathcal{Z}^n, \mathcal{K}^n) \in \mathcal{S}_c^\infty(\Delta_n(T), E^n) \times \mathbf{L}_W^2(\Delta_n(T), E^n) \times \mathbf{A}(\Delta_n(T), E^n)$ to $\mathbf{EQ}(\mathcal{H}^n, \mathfrak{f}^n)$.

For $k \in \{0, 1, \dots, n-1\}$. Assume there exists $(\mathcal{Y}^{k+1}, \mathcal{Z}^{k+1}, \mathcal{K}^{k+1}) \in \mathcal{S}_c^\infty(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{L}_W^2(\Delta_{k+1}(T), E^{k+1}) \times \mathbf{A}(\Delta_{k+1}(T), E^{k+1})$ satisfying $\mathbf{EQ}(\mathcal{H}^{k+1}, \mathfrak{f}^{k+1})$. Consider the truncated generator

$$\mathfrak{f}^{k,N}(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) = \inf_{\pi \in A^k} \mathbf{g}^k(\pi, t, -N \vee y, z, \boldsymbol{\theta}_k, \mathbf{e}_k).$$

Then there exists some constant $C_N > 0$, independent of $(\boldsymbol{\theta}_k, \mathbf{e}_k)$, such that $|\mathfrak{f}^{k,N}| \leq C_N(1 + z^2)$. Hence, there exists a solution $(\mathcal{Y}^{k,N}, \mathcal{Z}^{k,N}, \mathcal{K}^{k,N}) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ to $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^{k,N})$. By Assumption **(HB')**, $\mathcal{Y}^{k,N} \geq \mathcal{H}^k \geq -C$, where $C > 0$ is a constant independent of N and $(\boldsymbol{\theta}_k, \mathbf{e}_k)$. Therefore, for N large enough, $(\mathcal{Y}^{k,N}, \mathcal{Z}^{k,N}, \mathcal{K}^{k,N})$ also solves $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^k)$. \square

5.4.3. RBSDE characterization by verification theorem.

Theorem 5.8. *The value functions $(\mathfrak{V}^k)_{k=0}^n$ defined in (5.15) and (5.16), are given by*

$$(5.17) \quad \mathfrak{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = U(x - \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)),$$

for $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k \times E^k$, where $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k)_{k=0}^n \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_W^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ is a solution of the system of RBSDEs $\mathbf{EQ}(\mathcal{H}^k, \mathfrak{f}^k)$, $k = 0, \dots, n$. Moreover, there exists a saddle point $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ described by:

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \arg \min_{\pi \in A^k} \mathbf{g}^k(\pi, t, \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \mathcal{Z}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k), \boldsymbol{\theta}_k, \mathbf{e}_k),$$

for $t \in [\theta_k, T]$, and

$$(5.18) \quad \hat{\tau}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) := \inf \left\{ t \geq \theta_k : \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = \mathcal{H}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \right\},$$

for $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, a.s., $k = 0, \dots, n$. More specifically, for any admissible trading strategy $(\pi, \tau) \sim (\pi^k, \tau^k)_{k=0}^n$,

$$\mathbb{E} [U(\mathfrak{X}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n}] \leq \mathbb{E} [U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n}] \leq \mathbb{E} [U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_n}],$$

and similar inequalities hold for $k = 0, \dots, n-1$, where $\hat{\mathfrak{X}}^{k,x}$ is the wealth process under control $\hat{\pi}^k$, $k = 0, \dots, n$.

Proof. We follow the steps in the proof of Theorem 5.4.

Step 1: We will show for $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$,

$$(5.19) \quad U(x - \mathcal{Y}_{\theta_k}^k(\boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k)) \geq \mathfrak{V}^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k), \quad k = 0, \dots, n.$$

Let $(\mathcal{Y}^k, \mathcal{Z}^k, \mathcal{K}^k) \in \mathcal{S}_c^\infty(\Delta_k(T), E^k) \times \mathbf{L}_{W'}^2(\Delta_k(T), E^k) \times \mathbf{A}(\Delta_k(T), E^k)$ be a solution of the RBSDE system. For $\nu^k \in \mathcal{A}^k$, $\forall x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \mathbf{e}_k) \in \Delta_k(T) \times E^k$, define $(\xi^k)_{k=0}^n$ as:

$$(5.20) \quad \begin{aligned} \xi_t^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^k) &:= U(\mathfrak{X}_t^{k,x} - \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) \\ &+ \int_{\theta_k}^t \int_E U\left(\mathfrak{X}_r^{k,x} + \nu_r^k \cdot \gamma_r^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e) - \mathcal{Y}_r^{k+1}(\mathbf{e}_k, r, \mathbf{e}_k, e)\right) \eta(de) dr, \end{aligned}$$

for $k = 0, \dots, n-1$, and

$$(5.21) \quad \xi_t^n(x, \boldsymbol{\theta}_n, \mathbf{e}_n, \nu^n) := U(\mathfrak{X}_t^{n,x} - \mathcal{Y}_t^n(\boldsymbol{\theta}_n, \mathbf{e}_n)).$$

Applying Itô's formula, we obtain, for $k = 0, \dots, n$,

$$\begin{aligned} d\xi_t^k(x, \boldsymbol{\theta}_k, \mathbf{e}_k, \nu^n) &= pU(\mathfrak{X}_t^{k,x} - \mathcal{Y}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)) \left[(-\mathfrak{f}^k(t, \mathcal{Y}_t^k, \mathcal{Z}_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k) \right. \\ &\quad \left. + \mathfrak{g}^k(\nu_t^k, t, \mathcal{Y}_t^k, \mathcal{Z}_t^k, \boldsymbol{\theta}_k, \mathbf{e}_k)) dt - d\mathcal{K}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + (\mathcal{Z}_t^k - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \nu_t^k) \cdot dW_t \right], \end{aligned}$$

Define $\hat{\tau}^k$ as in (5.18), then $d\mathcal{K}_{t \wedge \hat{\tau}^k}^k = 0$, $\theta_k \leq t \leq T$. Therefore, $(\xi_{t \wedge \hat{\tau}^k}^k)_{\theta_k \leq t \leq T}$ is a local supermartingale. By introducing a localizing sequence of stopping times $(\rho_m)_m$, and then letting $m \rightarrow \infty$, we can show for $k = 0, \dots, n$,

$$\xi_{t \wedge \hat{\tau}^k}^k \geq \mathbb{E} [\xi_{s \wedge \hat{\tau}^k}^k | \mathcal{F}_t], \quad \theta_k \leq t \leq s \leq T.$$

In particular,

$$(5.22) \quad U(x - \mathcal{Y}_{\theta_n}^n(\boldsymbol{\theta}_n, \mathbf{e}_n)) = \xi_{\theta_n}^n \geq \mathbb{E} [\xi_{\hat{\tau}^n}^n | \mathcal{F}_{\theta_n}] = \mathbb{E} [U(\mathfrak{X}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n}].$$

Hence,

$$U(x - \mathcal{Y}_{\theta_k}^n(\boldsymbol{\theta}_k, \mathbf{e}_k)) \geq \text{ess inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E}[U(\mathfrak{X}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_k}].$$

for any $\nu^n \in \mathcal{A}^n$. So (5.19) follows for $k = n$. Similarly, it holds for $k = 0, \dots, n-1$.

Steps 2& 3: Similar to the proof of Theorem 5.4.

Step 4: We will show (5.17) holds and $(\pi, \tau) \sim (\hat{\pi}^k, \hat{\tau}^k)_{k=0}^n$ is a saddle point. Under the admissible control $\hat{\pi}^k$, the dynamics of $(\xi^k)_k$ defined in (5.20) and (5.21) are given by

$$d\xi_t^k(x, \boldsymbol{\theta}, \mathbf{e}, \hat{\pi}^k) = pU(\mathfrak{X}_t^{k,x} - \mathcal{Y}_t^k) \left[-d\mathcal{K}_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + (\mathcal{Z}_t^k - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \nu_t^k) \cdot dW_t \right],$$

for $k = 0, \dots, n$. By the uniform integrality of ξ_t^k , we know ξ_t^k is a sub-martingale. Consider when $k = n$. For any \mathbb{F} -stopping time τ^n valued in $[\theta_n, T]$,

$$(5.23) \quad U(x - \mathcal{Y}_{\theta_n}^n) \leq \mathbb{E}[U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{Y}_{\tau^n}^n) | \mathcal{F}_{\theta_n}] \leq \mathbb{E}[U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_n}],$$

Therefore, we have

$$U(x - \mathcal{Y}_{\theta_n}^n) \leq \operatorname{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_n} \right] \leq \operatorname{ess\,sup}_{\pi^n \in \mathcal{A}^n} \operatorname{ess\,inf}_{\tau^n \in \mathcal{T}^n} \mathbb{E} \left[U(\mathfrak{X}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_n} \right].$$

Now, the last equation along with (5.19) implies that (5.17) holds for $k = n$.

By the definition and admissibility of $\hat{\pi}^n$, we can show that under control $\hat{\pi}^n$, $\xi_{t \wedge \hat{\tau}^n}^n$ is a martingale. Thus from (5.23) we have

$$\begin{aligned} \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] &= \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} - \mathcal{Y}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] = U(x - \mathcal{Y}_{\theta_n}^n) \\ &\leq \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{Y}_{\tau^n}^n) | \mathcal{F}_{\theta_n} \right] \leq \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\tau^n}^{n,x} - \mathcal{H}_{\tau^n}^n) | \mathcal{F}_{\theta_n} \right]. \end{aligned}$$

And from (5.22) we have

$$\begin{aligned} \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] &= \mathbb{E} \left[U(\hat{\mathfrak{X}}_{\hat{\tau}^n}^{n,x} - \mathcal{Y}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] = U(x - \mathcal{Y}_{\theta_n}^n) \\ &\geq \mathbb{E} \left[U(\mathfrak{X}_{\hat{\tau}^n}^{n,x} - \mathcal{Y}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] = \mathbb{E} \left[U(\mathfrak{X}_{\hat{\tau}^n}^{n,x} - \mathcal{H}_{\hat{\tau}^n}^n) | \mathcal{F}_{\theta_n} \right] \end{aligned}$$

Thus, $(\hat{\pi}^n, \hat{\tau}^n)$ is a saddle point. Similarly, it can be shown that the corresponding conclusions hold for $k = 0, \dots, n-1$ using (5.16). \square

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